1. Computability, Complexity and Algorithms

Problem (Making bags of candy). The owner of the store CandyLand receives supplies every morning, and immediately sets the candy to be sold. This is done by creating identical bags, each of size S. The store receives n different types of candy, and a finite amount of each type.

Your input is: integer S, representing the size of each bag, and an array of integers `Type` of length n. Entry `Type`$[i]$ is the number of candy of type i. Note:

- Each bag must have exactly S pieces of candy, and the same number of each type.
- The objective is to maximize the number of bags the salesman can make.
- The optimal solution does not need to use all the candy, nor all the types of candy.
- Example: for $S = 2$ and `Type`$= [10, 12, 14]$ the answer is 12, as we can make these many bags with one piece of types 2 and 3 each.
- Example: for $S = 3$ and `Type`$= [10, 12, 14]$ the answer is 10, putting one piece of each type in every bag.

(a) (6 points) Design a polynomial time algorithm to find the maximal number of bags the salesman can make.

(b) (4 points) Suppose we also require that each bag must have at least k different types of candy. Design a polynomial time algorithm to find the maximal number of bags the salesman can make.

Explain why your algorithm works and analyze its running time in terms of the input size. You may assume that standard mathematical operations take $O(1)$.

2. Theory of Linear Inequalities

Problem. Given polytopes $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^{n+q}$ such that $P = proj_x(Q)$ be the projection of Q, i.e.

$$P = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^q, (x, y) \in Q\}.$$

Then show that

1. (5 points) Q has at least as many faces as P.
2. (3 points) The number of facets of Q is at least the logarithm (to base 2) of the number of faces of P.

3. (2 points) Consider the permutahedron $\Pi_n \subset \mathbb{R}^n$, the convex hull of the set S_n of the $n!$ vectors that can be obtained by permuting the entries of the vector $(1, 2, \ldots, n-1, n)$. It is easy to check that each permutation of $(1, 2, \ldots, n)$ gives a vertex of Π_n (You need not prove this fact). If U is a polytope such that $proj_x(U) = \Pi_n$ show that U must have at least $\Omega(n \log n)$ facets.

3. Graph Theory

Problem. Let G be a 4-connected graph and s_1, s_2, s_3, s_4 be four distinct vertices of G, and assume that $G - \{s_1, s_2, s_3, s_4\}$ contains a subdivision of K_6. Show that G contains two vertex disjoint paths, one from s_1 to s_3 and the other from s_2 to s_4.

4. Analysis of (Advanced) Algorithms

Problem. Let A be an $n \times m$ integer matrix. Consider the matrix

$$P = A(A^\top A)^+ A^\top \,,$$

where for any matrix B, B^+ denotes the pseudo-inverse of B. (If $B = UDV^T$ is the SVD of B, then $B^+ = VD^{-1}U^T$. Show that

1. $P^2 = P$.
2. The eigenvalues of P are all 0 or 1.
3. P is the orthogonal projection to the span of the columns of A (i.e., $Px = x$ if x is in the span of the columns of A and $Px = 0$ if it is orthogonal to the span.)
4. The leverage score of a row a_i of A

   $$\sigma_i = a_i(A^\top A)^+ a_i^\top$$

   satisfies $0 \le \sigma_i \le 1$.
5. $\sigma_i(A) = \left\|A(A^\top A)^+ A^\top e_i\right\|_2^2$.
6. Let

   $$\widetilde{\sigma}_i = \left\|RA(A^\top A)^+ A^\top e_i\right\|_2^2 \,,$$

   where R is a random $k \times n$ matrix with iid Gaussian entries from $N(0, \frac{1}{k})$. How well does $\widetilde{\sigma}_i$ approximate σ_i for every i? What should k be to guarantee a $1+\epsilon$ relative error approximation whith high probability?
7. Given an oracle for solving linear systems, how would you compute $\widetilde{\sigma}_i$ efficiently?

5. Combinatorial Optimization

Problem.

(i) (4 points) Let E be a finite set and $f : 2^E \to \mathbb{R}_+$ a submodular function with $f(\{e\}) \le 2$ for all $e \in E$. We have an oracle $\mathcal{O}_1$ to find the maximum cardinality set $X \subseteq E$ with $f(X) = 2|X|$, for any such given submodular function f (given by a value oracle). This problem is called the Polymatroid Matching Problem.

Disjoint Pairs Problem: Let $E_1, \ldots, E_k$ be pairwise disjoint unordered pairs and let $(E, \mathcal{F})$ be a matroid (given by an independence oracle), where $E = E_1 \cup E_2 \cup \ldots \cup E_k$. The

DISJOINT PAIRS PROBLEM is to find the maximum cardinality set $I \subseteq \{1, \ldots, k\}$ such that $\bigcup_{i \in I} E_i \in \mathcal{F}$.

Give a polynomial time algorithm (in the size of the ground set E) using oracle $\mathcal{O}_1$ (i.e., each call to the oracle is counted as a single step) to solve the DISJOINT PAIRS PROBLEM. In other words, reduce the DISJOINT PAIRS PROBLEM to the POLYMATROID MATCHING PROBLEM polynomially.

(ii) (6 points) Now assume that you have an oracle $\mathcal{O}_2$ to solve the DISJOINT PAIRS PROBLEM.

Consider a graph $G = (V, E)$ with each edge belonging to one color class: red, blue and green. We want to find the maximum cardinality acyclic subset of edges so that the number of edges of any single color does not exceed $\lceil n/3 \rceil$. Show that this is a matroid intersection problem (2 points).

Reduce matroid intersection to the DISJOINT PAIRS PROBLEM polynomially, i.e., give a polynomial time algorithm to find maximum cardinality set in the intersection of two matroids using the given oracle $\mathcal{O}_2$. (4 points). Assume oracle access to the rank functions of the matroids as well.

6. Probabilistic Methods

Problem. Suppose that there are k people in a lift at the ground floor, and that each wants to get off at a random floor of one of the n upper floors (to clarify: all people make their choices independently). Let X denote the number of lift stops, i.e., total number of distinct floors chosen by the k people. For $k = k(n)$ satisfying $k \to \infty$ and $k/n \to 0$ as $n \to \infty$, show that the expectation and variance of X satisfy $\mathbb{E}X \sim k$ and $\operatorname{Var} X = o((\mathbb{E}X)^2)$.

Hints:

- Working with the 'complement' $Y := n - X$ might be easier.
- Recall that $(1 - x)^r = 1 - rx + O((rx)^2)$ and $(1 - x)^r = 1 - rx + \binom{r}{2}x^2 + O((rx)^3)$ when $r \geq 2$ and $x \in [0, 1]$.