1. Computability, Complexity and Algorithms

Problem (dismantling a tree). You are given a weighted tree and your task is to dismantle it in the cheapest possible way. The only operation you are allowed is the following: you can erase any path such that all edges in the path have the same weight ω, and the price of doing this is ω. The tree is dismantled once you erased all its edges. Design an algorithm that returns the minimal cost to dismantle a given tree with weights.

Your input is a tree $T = (V, E)$ with vertex set $V := \{0, 1, \ldots, n-1\}$ and edge set $E := \{(u_1, v_1), (u_2, v_2), \ldots, (u_{n-1}, v_{n-1})\}$, where $u_i, v_i \in V$, for all i. You are also given the weight of each edge: let an integer $0 < w_i < K$ be the weight of edge (u_i, v_i). Your output should be a single integer denoting the minimal cost to dismantle the input tree. Explain why your algorithm is correct and analyze its running time in terms of n and K.

2. Theory of Linear Inequalities

Problem. Let $P = \{x \in \mathbf{R}^n : \sum_{i \in S} x_i - \sum_{i \notin S} x_i \leq 1\,, \quad \forall S \subseteq \{1, 2, \ldots, n\}\}$. Show that

1. (4 pts) e_i and $-e_i$ are vertices of P, where e_i is the ith standard vector in $\mathbf{R}^n$.
2. (3 pts) All the 2^n inequalities, $\sum_{i \in S} x_i - \sum_{i \notin S} x_i \leq 1$ (for each subset S), are facets of P.
3. (3 pts) Show that $P = conv(\{e_1, e_2, \ldots, e_n, -e_1, -e_2, \ldots, -e_n\})$.

3. Graph Theory

Problem. Let $k \geq 3, t \geq 3$ be positive integers and let G be a graph with clique number k. Show that if G does not contain $K_{1,t}$ as an induced subgraph then $\chi(G) < R(k, t)$, where $\chi(G)$ as usual denotes the chromatic number of G and $R(k, t)$ denotes the Ramsey number with respect to clique of size k and independent set of size t.

4. Analysis of Algorithms

Problem. Suppose that we have n unit squares $S_1, \ldots, S_n$ in the Euclidean plane, each of which has side length 1. They may not align with the axes and may overlap with each other. You are given the positions (coordinates of the four vertices) of all squares. Let $S = S_1 \cup \cdots \cup S_n$ be the *union* of these squares.

Part (a): Consider the following algorithm:
Pick i from $\{1, \ldots, n\}$ uniformly at random and then pick a point x from S_i uniformly at random. If $x \notin S_j$ for all $1 \leq j < i$, then the algorithm succeeds and outputs x; otherwise it fails.
Prove that this algorithm succeeds with probability at least $1/n$, and when it succeeds the point it outputs is from the uniform distribution over S.

Part (b): Give a randomized algorithm that approximately estimates the area of S. More specifically, for given $0 < \varepsilon < 1$, the output Z of your algorithm should satisfy

$$\mathbf{Pr}\left[(1-\varepsilon)|S| \leq Z \leq (1+\varepsilon)|S|\right] \geq 3/4 \qquad (*)$$

where $|S|$ denotes the area of S. The running time of your algorithm should be polynomial in n and $1/\varepsilon$.

State your algorithm and its running time, and include the analysis of your algorithm. You may assume that for every square S_i, in $O(1)$ time you can check membership (is $x \in S_i$?), and in $O(1)$ time you can generate a point uniformly at random from S_i. You may neglect all numerical issues caused by real numbers.

5. Combinatorial Optimization

Problem. Given a planar graph $G = (V, E)$ with weighted (multiple) edges, give a polynomial time algorithm to find the minimum-weight edge set M that can be removed such that the graph $H = (V, E \setminus M)$ is 2-colorable (i.e., every vertex can be colored with two colors, such that no two adjacent vertices have the same color).

6. Probabilistic methods

Problem. Given a finite set Γ, let Γ_p denote the random subset where each element $x \in \Gamma$ is included independently with probability p. Given any event $\mathcal{E}$ (a family of subsets of Γ), to avoid clutter we write $\Pr(\mathcal{E}) = \Pr(\Gamma_p \in \mathcal{E})$, as usual. Furthermore, we say that $\mathcal{E}$ is *increasing* if $\mathcal{X} \subseteq \mathcal{X}^+ \subseteq \Gamma$ and $\mathcal{X} \in \mathcal{E}$ imply $\mathcal{X}^+ \in \mathcal{E}$. Similarly, we say that $\mathcal{E}$ is *decreasing* if $\mathcal{X}^- \subseteq \mathcal{X}$ and $\mathcal{X} \in \mathcal{E}$ and imply $\mathcal{X}^- \in \mathcal{E}$.

Setup/What you may assume: Let $\mathcal{I}_1, \mathcal{I}_2, \ldots$ denote increasing events, and let $\mathcal{D}$ denote a decreasing event. In the following sub-tasks you may only assume *Harris inequality* (a special case of the FKG inequality), which states that any two increasing events $\mathcal{I}$ and $\mathcal{J}$ are positively correlated, i.e., satisfy $\Pr(\mathcal{I} \cap \mathcal{J}) \geq \Pr(\mathcal{I}) \Pr(\mathcal{J})$.

(a) Show that $\Pr(\mathcal{I}_1 \cap \cdots \cap \mathcal{I}_k) \geq \Pr(\mathcal{I}_1) \cdots \Pr(\mathcal{I}_k)$ for all integers $k \geq 1$.
(b) Show that $\Pr(\mathcal{I}_1 | \mathcal{D}) \leq \Pr(\mathcal{I}_1)$.
(c) Show that if $\mathcal{I}_1$ and $\mathcal{I}_2$ are independent, then $\Pr(\mathcal{I}_1 \mid \mathcal{I}_2 \cap \mathcal{D}) \leq \Pr(\mathcal{I}_1)$.