ACO Comprehensive Exam Spring 2022

Jan 7, 2022

1 Algorithms

Alice is about to enter the Queen's mirror room. The room is mapped as a $N \times N$ unit grid, and Alice can only move from one point to a neighboring point (two points are neighbors if they are at distance one). Mirrors are placed at certain points of the room. The room is completely dark, and the only way to see a path is to project a straight ray of light from the entrance. Light reflect off the mirrors. The problem is that the mirrors are magical and Alice must activate them in order for them to reflect light (at first, all mirrors are inactive). When light hits an active mirror, the light goes into the four directions of the grid, but if the mirror is inactive the ray of light passes straight through it.

Help Alice find a path to escape the room.

1. (4 points) You are given a natural number N and K integer points where mirrors are placed, $(x_1, y_1), (x_2, y_2), \ldots, (x_K, y_K)$, $0 \leq x_i, y_i \leq N - 1$; the location of the entrance A and the location of the exit B, both are points on the boundary of the room. You must output the minimal number of mirrors Alice needs to activate such that a ray of light starting at A will reach B, OR report that the task is impossible. Describe your algorithm and justify its correctness. Analyze its runtime.

2. (6 points) The Queen is feeling merciful, and opened M exits of the room. Help Alice find any one exit! As input, you are given natural numbers N, M; K points $(x_1, y_1), \ldots, (x_K, y_K)$ where mirrors are placed, $0 \leq x_i, y_i \leq N - 1$; the entrance A and exits $B_1, B_2, \ldots, B_M$, all points on the boundary of the room. You must output the minimal number of mirrors Alice needs to activate such that a ray of light out of A will reach B_i for at least one value of i, OR report that the task is impossible. Describe your algorithm and justify its correctness. Analyze its runtime.

2 Graph Theory

Let G be a connected graph with girth at least 11 and minimum degree at least k (sufficiently large if needed). Let X be a maximal subset of $V(G)$ satisfying $d(x_1, x_2) \geq 3$ for all distinct $x_1, x_2 \in X$. Here $d(x_1, x_2)$ denotes the distance between x_1 and x_2 in G.

1. (4 points) Show that $V(G)$ admits a partition V_x, $x \in X$, such that
   (a) $G[V_x]$, the subgraph of G induced by V_x, is a tree,
   (b) V_x contains x and all neighbors of x, and
   (c) all vertices in V_x are within distance 2 from x in G.
2. (3 points) Let H be obtained from G by contracting $G[V_x]$ for every $x \in X$. Show that the minimum degree of H is at least $k(k-1)$.
3. (3 points) Prove that H contains a subdivision of K_t for some t that is linear in k. (You may assume that $10s$-connected graphs are s-linked. A graph K is s-linked if, for all distinct vertices $u_1, \ldots, u_s, v_1, \ldots, v_s$ of K, K contains pairwise disjoint paths joining u_i to v_i for all $i \in [s]$.)

3 Linear Inequalities

Given a positive integer $t \geq 2$, consider $P := \{x \in R^2_+ : tx_1 + x_2 \leq 1 + t, -tx_1 + x_2 \leq 1\}$ and $S = P \cap \mathbb{Z}^2$.

1. (3 points). Describe all vertices of P and $conv(S)$.
2. (5 points). Show that if a polytope Q contains $\big((0,0),(0,1),(1,0),\big(\frac{1}{2}, 1 + \frac{k}{2}\big)\big)$ for any integer $k \geq 2$ then

$$\left((0,0),(0,1),(1,0),\left(\frac{1}{2}, 1 + \frac{(k-1)}{2}\right)\right) \in Q'$$

   where Q' is the Chvátal closure of Q.
3. (2 points) Show that the Chvátal rank of P is at least t.