ACO Comprehensive Exam Spring 2022

Jan 7, 2022

1 Algorithms

Alice is about to enter the Queen's mirror room. The room is mapped as a $N \times N$ unit grid, and Alice can only move from one point to a neighboring point (two points are neighbors if they are at distance one). Mirrors are placed at certain points of the room. The room is completely dark, and the only way to see a path is to project a straight ray of light from the entrance. Light reflect off the mirrors. The problem is that the mirrors are magical and Alice must activate them in order for them to reflect light (at first, all mirrors are inactive). When light hits an active mirror, the light goes into the four directions of the grid, but if the mirror is inactive the ray of light passes straight through it.

Help Alice find a path to escape the room.

1. (4 points) You are given a natural number N and K integer points where mirrors are placed, $(x_1, y_1), (x_2, y_2), \ldots, (x_K, y_K)$, $0 \leq x_i, y_i \leq N-1$; the location of the entrance A and the location of the exit B, both are points on the boundary of the room. You must output the minimal number of mirrors Alice needs to activate such that a ray of light starting at A will reach B, OR report that the task is impossible. Describe your algorithm and justify its correctness. Analyze its runtime.

2. (6 points) The Queen is feeling merciful, and opened M exits of the room. Help Alice find any one exit! As input, you are given natural numbers N, M; K points $(x_1, y_1), \ldots, (x_K, y_K)$ where mirrors are placed, $0 \leq x_i, y_i \leq N-1$; the entrance A and exits $B_1, B_2, \ldots, B_M$, all points on the boundary of the room. You must output the minimal number of mirrors Alice needs to activate such that a ray of light out of A will reach B_i for at least one value of i, OR report that the task is impossible. Describe your algorithm and justify its correctness. Analyze its runtime.

Solution

(a) Let's formulate this problem as an undirected graph problem.
Let the rows and the columns be the vertices ($2N$ total vertices). Show the vertex corresponding to row i ($1 \leq i \leq N$) by v_i and show the vertex corresponding to column j ($1 \leq j \leq N$) by v_{N+j}. Let the mirrors be the undirected edges. A mirror located in cell (i, j) connects vertex v_i to vertex v_{N+j}. Activating a mirror corresponds to taking the corresponding edge. Denote by v_A the vertex corresponding to row/column A and analogously define v_B. If we want to pass the light from row/column A, to row/column B, there should be a path from v_A to v_B in this graph. If we want to find the minimum number of the active mirrors, we just need to find the shortest path from v_A to v_B. Our graph has $O(N)$ vertices and at most $O(N^2)$ edges (worst case is when there is a mirror in every cell. This means we have a complete bipartite graph where every v_i is connected to every v_{N+j}, where $1 \leq i \leq N$ and $1 \leq j \leq N$). Use BFS starting from v_A and stop when v_B is discovered. We know that in an undirected and unweighted graph, BFS gives the shortest path. And we know that BFS runs in time $O(|V| + |E|)$, which is $O(N + N^2) = O(N^2)$ in our problem.

(b) Consider the graph from part **(a)**, add a vertex $\mathcal{O}$ and an edge from each B_i to it, for $1 \leq i \leq M$. In order for Alice to reach $\mathcal{O}$, it must reach at least one of the B_i. Hence, the shortest path from A to $\mathcal{O}$ yields the shortest path to one of the exits B_i, as desired. The runtime is $O(M + N^2) = O(N^2)$ since M is linear in N (since all exits are boundary points).

2 Graph Theory

Let G be a connected graph with girth at least 11 and minimum degree at least k (sufficiently large if needed). Let X be a maximal subset of $V(G)$ satisfying $d(x_1, x_2) \geq 3$ for all distinct $x_1, x_2 \in X$. Here $d(x_1, x_2)$ denotes the distance between x_1 and x_2 in G.

1. (4 points) Show that $V(G)$ admits a partition V_x, $x \in X$, such that
   (a) $G[V_x]$, the subgraph of G induced by V_x, is a tree,
   (b) V_x contains x and all neighbors of x, and
   (c) all vertices in V_x are within distance 2 from x in G.
2. (3 points) Let H be obtained from G by contracting $G[V_x]$ for every $x \in X$. Show that the minimum degree of H is at least $k(k-1)$.
3. (3 points) Prove that H contains a subdivision of K_t for some t that is linear in k. (You may assume that $10s$-connected graphs are s-linked. A graph K is s-linked if, for all distinct vertices $u_1, \ldots, u_s, v_1, \ldots, v_s$ of K, K contains pairwise disjoint paths joining u_i to v_i for all $i \in [s]$.)

Solution. Let $X_i = \{v \in V(G) : d(v, X) = i\}$, where $d(v, X)$ is the distance in G between v and X. By the maximality of X, $X_i = \emptyset$ for $i \geq 3$. Construct V_x as follows. First, put x in V_x for each $x \in X$. For each $v \in X_1$, there is a unique $x \in X$ such that $v \in N(x)$ (by definition of X), and add v into this V_x. Thus, $N(x) \subseteq V_x$ for $x \in X$. Now let $v \in X_2$. Then there exists some $x \in X$ such that v is adjacent to $N(x)$, and choose one such x and add v to this V_x.

The above process produces a partition of $V(G)$ into V_x, $x \in X$, such that for $x \in X$, we have $x \in V_x$, $N(x) \subseteq V_x$, and every vertex in V_x is within distance 2 from x. Moreover, by construction, $G[V_x]$ is connected and, since G has girth at least 11, $G[V_x]$ is a tree. So we have (1).

To prove (2), we observe that each $G[V_x]$ has at least k leaves and each leaf has at least $k-1$ neighbors in G outside V_x. Since the girth of G is at least 11, we see that G has at most one edge between V_{x_1} and V_{x_2} for all distinct $x_1, x_2 \in X$. Thus, the minimum degree of H is at least $k(k-1)$.

By (2) and a result of Mader, H contains a subgraph, say H', that is $\lfloor k(k-1)/4 \rfloor$-connected. Write $10s = \lfloor k(k-1)/4 \rfloor$. Then H' is $\lfloor s \rfloor$-linked. Let t be the maximal integer such that $\binom{t}{2} \leq \lfloor s \rfloor$ (so t is linear in k). Choose t vertices in H and make $(t-1)$ new copies for each such vertex. Now find $\binom{t}{2}$ disjoint paths linking those vertices appropriately. Identifying the copies back to the t vertices in H, we obtain a subdivision of K_t in H.

3 Linear Inequalities

Given a positive integer $t \geq 2$, consider $P := \{x \in R^2_+ : tx_1 + x_2 \leq 1 + t, -tx_1 + x_2 \leq 1\}$ and $S = P \cap \mathbb{Z}^2$.

1. (3 points). Describe all vertices of P and $conv(S)$.
2. (5 points). Show that if a polytope Q contains $\left((0,0),(0,1),(1,0),(\frac{1}{2}, 1 + \frac{k}{2})\right)$ for any integer $k \geq 2$ then

$$\left((0,0),(0,1),(1,0),\left(\frac{1}{2}, 1 + \frac{(k-1)}{2}\right)\right) \in Q'$$

   where Q' is the Chvátal closure of Q.

3. (2 points) Show that the Chvátal rank of P is at least t.

(Solution:(a)) Observe that $P = conv\left((0,0)(0,1),(1+\frac{1}{t},0),(\frac{1}{2},1+\frac{t}{2})\right)$ and $conv(S) = ((0,0),(0,1),(1,0),($
(b). We show the following statement for any integer $k \geq 1$. If

$$\left((0,0),(0,1),(1,0),\left(\frac{1}{2}, 1 + \frac{k}{2}\right)\right) \in Q$$

then

$$\left((0,0),(0,1),(1,0),\left(\frac{1}{2}, 1 + \frac{(k-1)}{2}\right)\right) \in Q'$$

where Q' is the Chvátal closure of Q.
To see the claim, consider any valid inequality $a_1x_1 + a_2x_2 \leq b$ for integer a_1, a_2 that is valid for Q. Clearly, all integer points, i.e.,

$$((0,0),(0,1),(1,0))$$

are valid for the C-G inequality $a_1x_1 + a_2x_2 \leq \lfloor b \rfloor$. It is enough to show that $\left(\frac{1}{2}, 1 + \frac{(k-1)}{2}\right)$ is valid for the C-G inequality. Since $a_1x_1 + a_2x_2 \leq b$ is valid for Q, we must have

$$\begin{aligned} 0 &\leq b & (1)\\ a_2 &\leq b & (2)\\ a_1 &\leq b & (3)\\ \frac{a_1}{2} + a_2 + \frac{k}{2}a_2 &\leq b & (4) \end{aligned}$$

We need to show that $\frac{a_1}{2} + a_2 + \frac{k-1}{2}a_2 \leq \lfloor b \rfloor$.
Case 1. $a_2 > 0$. Then

$$\frac{a_1}{2} + a_2 + \frac{k-1}{2}a_2 = \frac{a_1}{2} + a_2 + \frac{k}{2}a_2 - \frac{1}{2}a_2 \leq b - \frac{1}{2}a_2 \leq b - \frac{1}{2}.$$

Since the LHS is a half-integral, it must be at most $\lfloor b \rfloor$ as required.
Case 2. $a_2 \leq 0$. We have $a_1 \leq b$ and $a_1 \in \mathbb{Z}$ and therefore, $a_1 \leq \lfloor b \rfloor$. Thus

$$\frac{a_1}{2} + (1 + \frac{k-1}{2})a_2 \leq a_1 + 0 \leq \lfloor b \rfloor.$$

In both cases, we are done as required.
(c). The lower bound on Chvátal rank of P follows from induction and observation that $(\frac{1}{2}, 1 + \frac{1}{2}) \notin conv(S)$.