ACO Comprehensive Exam Fall 2022

Aug 17, 2022

1 Design and Analysis of Algorithms

In a *combinatorial auction* there is a set N of $n = |N|$ bidders and a set M of $m = |M|$ items. Bidder $i \in N$ has a *monotone* valuation $v_i(\cdot)$ where $v_i(S)$ is their value for item set $S \subseteq M$ (here "monotone" means that $v_i(S \cup T) \geq v_i(S)$ for all $S, T \subseteq M$). The goal of this problem is to find a disjoint set of subsets where bidder i gets subset A_i of items ($A_i \cap A_k = \emptyset$ for $i \neq k$) to maximize the total *welfare* $\sum_{i \in N} v_i(A_i)$.

1. (Configuration LP, 2 points) Prove that the value of the following linear program (called the *configuration LP*) gives an upper bound on the total welfare of the optimal allocation.

$$\begin{aligned} \max \quad & \sum_{i \in N} \sum_{S \subseteq M} v_i(S) \cdot x_{i,S} \\ \text{s.t.} \quad & \forall i \in N, \quad \sum_{S \subseteq M} x_{i,S} = 1 \\ & \forall j \in M, \quad \sum_{S \ni j} \sum_{i \in N} x_{i,S} \leq 1 \\ & \forall i \in N, \forall S \subseteq M, \quad x_{i,S} \geq 0 \end{aligned}$$

2. (XOS Function, 3 points) A monotone set function $v(\cdot) : 2^M \to \mathbb{R}_{\geq 0}$ is called an *XOS function* if there exist monotone linear set functions $a_k(\cdot) : 2^M \to \mathbb{R}_{\geq 0}$ s.t. for all $S \subseteq M$ we have $v(S) = \max_k a_k(S)$ (i.e., $v(\cdot)$ can be written as the maximum of linear functions where a function $a_k(\cdot)$ is linear if it satisfies $a_k(S \cup T) = a_k(S) + a_k(T)$ for all disjoint $S, T \subseteq M$). Given a set $S \subseteq M$, prove that if we select a random subset $R \subseteq S$ from a probability distribution that contains each item in S with probability at least p (different items could be correlated), then the expected value of $v(R)$ is at least $p \cdot v(S)$.

3. (Rounding) Suppose we are given an optimal (fractional) solution $x^*_{i,S}$ to the configuration LP[1]. To "round" this fractional solution to integral allocations A_i, each bidder

[1] This can be computed in polynomial time using a "demand oracle" but we will assume that it is given.

$i \in N$ first chooses a random *tentative* item set T_i independent of other bidders, where $T_i = S$ with probability $x^*_{i,S}$ (the LP constraint $\sum_{S \subseteq M} x^*_{i,S} = 1$ ensures that this is a valid probability distribution). Since in this tentative allocation an item j might appear in multiple tentative sets, in the final allocation $\{A_i\}_i$ we allocate each item $j \in M$ to one of the tentative bidders (i.e., bidders i with $j \in T_i$) chosen uniformly at random.

(a) (3 points) Prove that conditioned on T_i, bidder i receives each item $j \in T_i$ with at least a constant probability, where the probability is taken over the random tentative sets T_k chosen by other bidders $k \neq i$.

(b) (2 points) Using (2), prove that if all valuations v_i are monotone XOS then the expected welfare of this rounded solution is at least a constant fraction of the optimal LP value $\sum_{i \in N} \sum_{S \subseteq M} v_i(S) \cdot x^*_{i,S}$, and so we get a constant factor approximation to the optimal welfare.

2 Combinatorial Optimization

Let $\mathcal{M} = (U, \mathcal{I})$ be a matroid with rank function $r : 2^U \to \mathbb{R}$ and let B and B' be two disjoint bases of $\mathcal{M}$. Let Y_1 and Y_2 be a partition of B. The problem is to prove the following statement:

There exists a partition Z_1 and Z_2 of B' such that $Y_1 \cup Z_1$ and $Y_2 \cup Z_2$ are both bases of $\mathcal{M}$.

To show this statement, prove the following steps (or give an alternative direct proof).

1. (1 point) We can assume without loss of generality that $U = B \cup B'$.
2. (2 points) Let $\mathcal{M}_1 = (\mathcal{M} \setminus Y_1)/Y_2$ and $\mathcal{M}_2 = (\mathcal{M}^\star \setminus Y_1)/Y_2$. Here $\mathcal{M}^\star$ is the dual matroid of $\mathcal{M}$ and $\mathcal{M}/Y$ denotes the matroid obtained by contracting elements in Y. What are the rank functions of $\mathcal{M}_1$ and $\mathcal{M}_2$ and, in particular, what are the ranks of these matroids?
3. (5 points) Show that there is a common independent set Z of size $|Y_1|$ of both these matroids.
4. (2 points) Show that $Z_2 = Z$ suffices to prove the statement.

3 Probabilistic Combinatorics

(10 points)
For a graph G, let $\mathsf{maxcut}(G)$ denote the maximum number of edges in a cut in G. Let $G \sim \mathbb{G}(n,p)$ be the Erdős–Rényi random graph with edge probability $p = p(n) \in [0,1]$ (so the edge probability is a function of n). Show that

$$\left| \mathbb{E}[\mathsf{maxcut}(G)] - \frac{pn^2}{4} \right| = O(\sqrt{p}\, n^{3/2}).$$

(The implied constants in the big-O notation should not depend on p.)
Remark. You may receive partial credit if you prove the result for only some range of values of p.