# ACO Comprehensive Exam Part I Fall 2023

Aug 18, 2023

## 1 Graduate Algorithms

Given a connected, undirected, weighted graph $G = (V, E)$, a *maximum spanning tree* is a spanning tree of $G$ of maximum total weight.

(a) (2 points) Given a connected, undirected, weighted graph $G = (V, E)$, with all edge weights distinct, prove the following *cycle property*:

For any cycle $\mathcal{C}$ of $G$, the edge of minimum weight in $\mathcal{C}$ does not belong to the maximum spanning tree.

(b) (3 points) Define the capacity of a path as the weight of the minimum edge in it. A bottleneck path between two vertices $u$ and $v$ is a path of maximum capacity among all paths connecting them. Design an algorithm that takes a weighted graph and pair of vertices $u, v$ as input and outputs a bottleneck path between $u$ and $v$. Analyze the runtime of your algorithm.

(c) (3 points) A network consists of a directed graph with nonnegative integer *capacities* $c(e)$ for every edge, along with a source vertex $s$ and a sink vertex $t$. Given a network $\{G = (V, E); s, t \in V; \{c(e)\}_{e \in E}\}$, a flow is a function $f : E \to \mathbb{R}$ such that

  (i) $0 \leq f(e) \leq c(e)$, for all $e \in E$

  (ii) $\sum_{u \in V} f(uv) = \sum_{w \in V} f(vw)$ for all $v \in V \setminus \{s, t\}$

The size of a flow is defined as $s(f) = \sum_{w \in V} f(sw)$ (i.e., the flow out of the source). Recall the classical approach to find a flow of maximum size: find an *augmenting path* in the residual network and update the flow along this path. Repeat this procedure until there is no such paths. Using your procedure from part (b) to find an augmenting path, i.e., using a bottleneck $(s, t)$-path in the residual graph as the augmenting path, show that if the current flow is $f$ and $f^*$ is any maximum flow, then the max capacity path has value at least $\frac{s(f^*) - s(f)}{m}$, where $m = |E|$.

(d) (2 points) Use part (c) to derive a polynomial-time algorithm to find a Maximum Flow in the given network.

## 2 Linear Inequalities

1. Given a polyhedron $P \subseteq \mathbb{R}^n$, let

$$P' = \left\{x \in \mathbb{R}^n : c^T x \leq \lfloor \delta \rfloor \ \forall (c, \delta) \in \mathbb{Z}^n \times \mathbb{R} \text{ s.t. } c^T x \leq \delta \text{ is valid for } P\right\}$$

denote the Chvátal-Gomory closure of $P$ and let $P_I = conv(P \cap \mathbb{Z}^n)$. For any integer $k \geq 0$, we also let $P^{(k)}$ denote the rank-$k$ C-G-closure of $P$, defined as $P^{(0)} = P$, $P^{(1)} = P'$ and $P^{(k)} = \left(P^{(k-1)}\right)'$ for any $k \geq 2$. The C-G rank of $P$ is the minimum integer $k$ such that $P^{(k)} = P_I$.

Given a graph $G = (V, E)$, let

$$Q(G) = conv(\{\chi(F) : (V, F) \text{ is a connected spanning subgraph of } G\})$$

be the convex hull of connected spanning subgraphs where $\chi(F) \in \{0, 1\}^E$ is the indicator vector for edge set $F$. You may use the fact that

$$Q(G) = \{x \in R^E : x(\delta_G(\pi)) \geq |\pi| - 1 \ \forall \text{ partitions } \pi \text{ of } V, 0 \leq x \leq 1\}$$

where $\delta_G(\pi)$ denotes the set of edges with endpoints in different parts of the partition $\pi$ and $|\pi|$ denotes the number of parts of $\pi$. Also, define the the cut-formulation

$$R(G) = \{x \in \mathbb{R}^E_+ : x(\delta(S)) \geq 1 \ \forall \ \emptyset \subsetneq S \subsetneq V, 0 \leq x \leq 1\}$$

where $\delta(S)$ denotes the set of edges with exactly one endpoint in $S$.

(a) (1 point) Show that $R(G)_I = Q(G)$.

(b) (2 points) Show that there exist graphs where C-G rank of $R(G)$ is at least 1.

(c) (6 points) Show that for every integer $k$, $R(G)^{(k)}$ satisfies all inequalities of $Q(G)$ for partitions of size up to $k + 2$, i.e., show that

$$R(G)^{(k)} \subseteq \{x \in R^E : x(\delta_G(\pi)) \geq |\pi| - 1 \ \forall \text{ partitions } \pi \text{ of } V \text{ s.t.} |\pi| \leq k+2, 0 \leq x \leq 1\}$$

(d) (1 point) What does the above part imply about the C-G rank of $R(G)$.

# ACO Comprehensive Exam Part II Fall 2023

Aug 17, 2023

## 1 Design and Analysis of Algorithms

- (4 points) Let $g : \mathbb{R}^n \to \mathbb{R}_+$ be a nonnegative $L$-Lipschitz function whose support is a convex body $K$ of Euclidean diameter $D$. Consider the following algorithm to estimate the integral of $g$ over $K$:

  1. Sample $k$ uniform random points from $K$ and compute the average value of $g(x)$ over the sample: $\mu_g = (1/k) \sum_{i=1}^{k} g(x_i)$.
  2. Output $\mu_g \cdot \text{Vol}(K)$

  What is the expected output of the algorithm, with a single random sample? Assuming access to oracles for sampling $K$ uniformly, and for its volume, how many samples $k$ suffice to get a multiplicative $(1+\epsilon)$-error approximation to the desired integral $\int_K g(x)\,dx$ with probability at least 3/4?

- (4 points) Now consider the problem of sampling from the density proportional to $g(x) = e^{-f(x)}$ restricted to a convex body $K$. Suppose you are given a membership oracle for $K$, i.e., numbers $r, R$ and a point $x_0$ s.t. $x_0 + rB_n \subseteq K \subseteq RB_n$, and an oracle for evaluating $f$ at any point $x \in K$. Consider the following algorithm:

  1. Sample $(x, t)$ with density proportional to $e^{-t}$ restricted to the set $\{(x,t) : x \in K, f(x) \leq t\}$
  2. Output $x$.

  Show how to use the ball walk to implement Step 1. Then show that the marginal density of $x$ is proportional to $e^{-f(x)}\chi_K(x)$. Give an interpretation of this algorithm as a reduction. You are not required to provide a convergence guarantee in this part.

- (2 points) What is the complexity status of sampling from the density proportional to $e^{-f(x)}$ restricted to $K$ when $f$ is convex, given by a function oracle and $\max_K f(x) - \min_K f(x) \leq \beta$; $K$ is convex, given by an $(r, R)$-membership oracle and a point $x_0$ s.t. $x_0 + rB_n \subseteq K \subseteq RB_n$ as above; and no other assumptions?

## 2 Combinatorial Optimization

Let $G = (V, E)$ be a undirected graph. Suppose that $E(S) \leq 2|S| - 2$ for all $S \subseteq V$ where $E(S) := \{\{u, v\} \in E : u, v \in S\}$ is the set of edges with both endpoints in $S$. Moreover, suppose that for some integer $k \geq 2$, $G$ contains a spanning tree $T$ such that the degree in $T$ of each vertex $v \in V$ is at most $k$. In the following, we will see how to find a spanning tree $H$, in *polynomial time*, such that degree of each vertex is at most $k + 2$.

1. (4 points) Show that there exists an orientation $D = (V, A)$ of $G = (V, E)$ such that each vertex $v \in V$ has in-degree at most two in $D$.

2. (2 points) For any set of edges $F \subseteq E$, we let $\vec{F}$ denote the corresponding oriented arcs in $D$. Show that if $H$ is a spanning tree of $G$ such that $\vec{H} \subseteq A$ and every vertex $v \in V$ has out-degree at most $k$ in $\vec{H}$, then degree of each vertex $v \in V$ is at most $k+2$ in $H$.

3. (4 points) Show that one can find in polynomial time, a spanning tree $H$ of $G$ such that degree of each vertex $v \in V$ in $H$ is at most $k + 2$. You may use part (b) and the fact that there is a polynomial-time algorithm for matroid intersection.

# 3 Probabilistic Combinatorics

**Problem.** Suppose we throw $n$ balls into $n$ bins independently and uniformly at random. Let $X$ be the random variable equal to the number of bins that remain empty.

$(a)$ (5 points) Prove that $\mathrm{Var}[X] \leq \mathbb{E}[X]$.

$(b)$ Prove one of the following upper bounds for all $\lambda > 0$:

$$\Pr\left[|X - \mathbb{E}[X]| > \lambda\sqrt{n}\right] \leq \begin{cases} e^{-1}\lambda^{-2} & \text{(partial credit, 2 points),} \\ \\ 2e^{-\lambda^2/2} & \text{(full credit, 5 points).} \end{cases}$$