1. Graph Theory

Prove that there exist no simple planar triangulation T and two distinct adjacent vertices $x, y \in V(T)$ such that x and y are the only vertices of T of odd degree. Do not use the Four-Color Theorem. (Simple means no loops or parallel edges.)

Solution: Let G be obtained from T by deleting the edge xy. Then every vertex of G has even degree, and hence the dual is bipartite. Thus the faces of G may be colored using colors 1 and 2 such that every edge is incident with a face of either color. Let a be the number of faces colored 1, and let b be the number of faces colored 2. From the symmetry we may assume that the unique face of G of size four is colored 1. Since every face of G colored 2 is a triangle, we deduce that $|E(G)| = 3b$. Since every face of G colored 1 is a triangle, except for precisely one face of size four, we deduce that $|E(G)| = 3(a-1)+4$. But the equation $3a + 1 = 3b$ has no integral solution, and hence no such triangulation exists.

2. Probability

Let $X, X_1, X_2, \dots$ be independent identically distributed random variables. Denote

$$S_n := X_1 + \cdots + X_n.$$

(a) If $X \geq 0$ a.s. and $\mathbb{E}X = +\infty$, then

$$\frac{S_n}{n} \to +\infty \text{ as } n \to \infty \text{ a.s.}$$

(b) If $\mathbb{E}|X| = +\infty$, then

$$\limsup_{n\to\infty} \frac{|S_n|}{n} = +\infty \text{ a.s.}$$

Solution: (a) Denote $X_j^c = X_j I(X_j \leq c),\ c > 0$. Then, by the SLLN, for all $c > 0$,

$$\frac{S_n}{n} \geq \frac{X_1^c + \cdots + X_n^c}{n} \to \mathbb{E}X_1^c \text{ as } n \to \infty \text{ a.s.}$$

Hence

$$\liminf_{n\to\infty} \frac{S_n}{n} \geq \sup_{c>0} \mathbb{E}X_1^c = \mathbb{E}X = +\infty \text{ a.s.},$$

implying (a).
(b) We have

$$+\infty = \mathbb{E}|X| = \int_0^{+\infty} \mathbb{P}\{|X| \geq t\} dt \leq c \sum_{n=0}^{\infty} \mathbb{P}\{|X| \geq cn\}.$$

Hence, for all $c > 0$,

$$\sum_{n=0}^{\infty} \mathbb{P}\{|X_n| \geq cn\} = \sum_{n=0}^{\infty} \mathbb{P}\{|X| \geq cn\} = +\infty.$$

By Borel-Cantelli lemma, this implies that for all $c > 0$ with probability 1

$$|X_n| \geq cn \text{ i.o..}$$

Therefore,

$$\limsup_{n\to\infty} \frac{|X_n|}{n} = +\infty \text{ a.s.} \quad (*)$$

It follows from the zero-one law that the probability of the event

$$\left\{ \limsup_{n\to\infty} \frac{|S_n|}{n} = +\infty \right\}$$

is either 0 or 1. If it is equal to 0, then with probability 1 the sequence $\{S_n/n\}$ is bounded implying that the same is true for the sequence $\{S_{n-1}/n\}$, and, as a consequence, for $\{X_n/n\}$. This contradicts $(*)$. Therefore

$$\limsup_{n\to\infty} \frac{|S_n|}{n} = +\infty$$

with probability 1.

3. Analysis of Algorithms

Suppose we would like to find a collection of matchings such that every edge of the graph is a member of (at least) one of the matchings we selected. The goal is to pick the minimum number of matchings for our collection. Give an $O(\log n)$-approximation for this problem that runs in polynomial time, where n is the number of vertices of the input graph.

Solution: Let the m edges of the input graph be the elements of our universe, and let each matching of the graph be a subset of this universe. The minimum number of matchings for our collection is then just a minimum set cover.

The greedy algorithm gives a $(\log m + 1)$-approximation algorithm for set cover that proceeds by iteratively choosing the set that covers the maximum number of elements that have not yet been covered. We can implement this efficiently by finding a maximum matching on the thus far unvisited edges, taking $O(n^2 m)$ time to find the matching. Since we remove at least one edge in each step, the total running time is at most $O(n^2 m^2)$ and this yields an $O(\log n)$-approximation algorithm.

4. Theory of Linear Inequalities

Let $C \subseteq R^n$ be a finitely-generated cone of full dimension and let H be an integral Hilbert basis for C. Suppose $w^T x \geq 0$ is a facet-defining inequality for C such that the components of w are relatively prime integers. Show that there exists a vector $h \in H$ such that $w^T h = 1$.

Solution available upon request.

5. Combinatorial Optimization

Consider an assignment problem (AP)

$$\begin{array}{lll} \max & \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij} & \\ & \sum_{j \in N} x_{ij} = 1 & \text{for } i \in N \\ & \sum_{i \in N} x_{ij} = 1 & \text{for } j \in N \\ & x_{ij} = 0 \text{ or } 1 & \text{for } i, j \in N \end{array}$$

where $N = \{1, \ldots, n\}$.

Note: You should be able to answer (1)-(5) quickly, which is preliminary to (6). (6) is the part of this question that counts the most.

(1) State a property of the constraint matrix from which it can be deduced that all of the extreme points of the LP relaxation are integral.

(2) Let $x(I, J) = \sum_{i \in I} \sum_{j \in J} x_{ij}$ where $\emptyset \subset I, J \subseteq N$. Suppose $|I| + |J| = n + k, k \geq 1$. Prove that if x is a feasible solution, then $x(I, J) \geq k$.

(3) Now consider a constrained AP called CAP where we require $x_{2k-1,2k-1} - x_{2k,2k} = 0$ for $k = 1, \ldots, m$. Suppose we have a fractional point say

$$x = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix} \text{ for } n = 3, m = 1.$$

What is required of such a point to conclude that the LP relaxation of CAP is not integral? Does this point x suffice?

(4) Let $n = 5, m = 2$ in CAP. Prove that

$$x_{11} - (x_{34} + x_{43} + x_{44}) \leq 0$$

is a valid inequality.

(5) Given $\emptyset \subset I, J \subseteq N$ with $|I| + |J| = n - 1$, let

$$K = \{2r - 1 : \{2r - 1, 2r\} \subseteq I \cap J, r \leq m\}$$
$$\hat{K} = \{2r - 1 : \{2r - 1, 2r\} \subseteq (N \backslash I) \cap (N \backslash J), r \leq m\}.$$

Show that the inequality in (4) is of the form

$$\sum_{i \in K \cup \hat{K}} x_{ii} - x(I, J) \leq 0 \qquad (*)$$

with $I = J = \{3, 4\}$.

(6) Prove in general that $(*)$ is a valid inequality for CAP if $|I| + |J| = n - 1$ and $|\hat{K}| \geq 1$.

(7) Describe how you would prove that $(*)$ with the condition given in (6) is a facet of the convex hull of CAP. An actual proof is not required.

(8) Give the form of $(*)$ with the condition given in (6) for $m = 1$. In this case, the assignment constraints, side constraints, nonnegativity and $(*)$ give the convex hull. Describe how you could prove this. An actual proof is not required.

(9) Suggest an idea for separating $(*)$ with $m = 1$.

Solution not yet available.

6. Algebra

Suppose G is a group of order 255. Prove that G is cyclic. (Hint: First show G has a normal subgroup of order 17 and that G has a normal cyclic subgroup of order 85.)

Solution: Let n_{17} be the number of Sylow 17 subgroup of G. We know n_{17} divides 15 and is congruent to 1 mod 17. Thus $n_{17} = 1$ and there is a unique Sylow 17 subgroup, call it P. Now G/P is a group of order 15. If we look at the number of Sylow 5 subgroups of G/P we see this number must divide 3 and be congruent to 1 mod 5. Thus there is just one such subgroup Q' and it must be normal. By the fourth isomorphism theorem we know there is a subgroup Q in G such that $Q/P = Q'$ in G/P. This subgroup will be normal and of order 85. (You could alternately argue that if P' was any Sylow 5 subgroup of G then PP' is a subgroup of G of order 85 and since its index is the smallest prime dividing the order of G we can conclude it is normal.) Clearly $P \leq Q$ is normal and since $5+1, 10+1$ and $15+1$ does not divide 17 the Sylow 5 subgroup R of Q is normal in Q. So $Q = PR, P \cap R = \{1\}$ and P and Q are both normal in Q. Thus $Q = P \times Q$ and since P and Q have prime orders we know they are cyclic. Thus $Q = (\mathbb{Z}/17\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z}) \cong \mathbb{Z}/85\mathbb{Z}$ is a cyclic subgroup of G. Now if n_3 is the number of Sylow 3 subgroups then n_3 divides 85 and is congruent to 1 mod 3. The only way for this to happen is for $n_3 = 1$ (since 4, 7, 10, 13, 16 and 19 do not divide 85). Thus the Sylow 3 subgroup S of G is normal in G. Since $G = QS, Q \cap S = \{1\}$ and Q and S are normal subgroups of G we see $G = Q \times S = \mathbb{Z}/255\mathbb{Z}$.

7. Randomized Algorithms

Recall for a pair of distributions μ and ν on a finite set Ω, their variation distance is

$$d_{\mathrm{TV}}(\mu,\nu) = \frac{1}{2}\sum_{z\in\Omega}|\mu(z)-\nu(z)|$$

Consider an ergodic Markov chain on state space Ω, transition matrix P and unique stationary distribution π. Let $P^t(x,\cdot)$ denote the t-step distribution of the Markov chain starting from $x \in \Omega$. Recall the mixing time is defined to be

$$T(\epsilon) = \max_{x\in\Omega} T_x(\epsilon)$$

where

$$T_x(\epsilon) = \min\left\{t : d_{\mathrm{TV}}(P^t(x,\cdot),\pi) \le \epsilon\right\}$$

For the purposes of this problem we consider the following notion of intersection time. For $x, y \in \Omega$, define their intersection time as

$$T^*_{x,y} = \min\left\{t : d_{\mathrm{TV}}\left(P^t(x,\cdot), P^t(y,\cdot)\right) \le 1/2\right\}$$

and let

$$T^* = \max_{x,y\in\Omega} T^*_{x,y}$$

Prove that $T(\epsilon) \le T^*\lceil\log(1/\epsilon)\rceil$, where the log is base 2.

Solution. We will prove the inequality using the coupling technique. For $x \in \Omega$, consider 2 copies (X_t) and (Y_t) of the Markov chain of interest where the initial states are $X_0 = x$ and Y_0 is chosen according to π. Thus Y_t is distributed according to π for all $t \ge 0$.

The coupling is defined in segments of T^* steps. From X_t, Y_t, we couple the 2 chains as follows:

1. Run X_t for T^* steps.
2. If $X_t = Y_t$, then we use the identity coupling for these T^* steps, i.e., let $Y_{t+1} = X_{t+1}, \ldots, Y_{t+T^*} = X_{t+T^*}$.
3. Otherwise, independently of what happened for X_t, run Y_t for T^* steps.
4. Repeat the process from X_{t+T^*}, Y_{t+T^*}.

By the definition of T^*, from any 2 initial states, if the 2 chains run independently for T^* steps, then with probability $\ge 1/2$ the 2 chains reach the same state. Thus, for any integer $i \ge 1$, for all $x', y' \in \Omega$, we have

$$Pr(X_{iT^*} \ne Y_{iT^*} | X_{(i-1)T^*} = x', Y_{(i-1)T^*} = y') \le 1/2$$

The coupling is defined so that if the 2 chains reach the same state at the end of a round of T^* steps then they always stay in the same state, therefore we have that

$$Pr(X_{iT^*} \ne Y_{iT^*}) \le (1/2)^i$$

By the coupling lemma, since Y_t is distributed according to π we have for $t = iT^*$:

$$d_{\mathrm{TV}}(P^t(x,\cdot),\pi) \le (1/2)^i$$

Setting $i = \lceil\log(1/\epsilon)\rceil$, and since the above holds for all initial states x, this proves the inequality.