1. Computability, Complexity and Algorithms

(a) (5 points) Let $G(V, E)$ be an undirected graph. A Hamilton Path in G is a path of legth $(|V| - 1)$, while a Hamilton Cycle in G is a cycle of length $|V|$. HC is the problem of deciding if a graph has a Hamilton Cycle, while HP is the problem of deciding if a graph has a Hamilton Path. We know that HC is NP-complete. Show that HP is NP-complete.

(b) (5 points) Let G be a connected undirected graph with at least 3 vertices. Let G^3 be the graph obtained by connecting all pairs of vertices that are connected by a path in G of length at most 3. Show that for all graphs G^3 as above, HC is in P. (Hint: Show that G^3 always has a Hamilton Cycle.)

2. Analysis of Algorithms

Consider the following algorithm for the weighted vertex cover problem, where $w(v)$ is the weight of vertex v. Initially $t(v) := w(v)$ for all vertices. When $t(v)$ drops to 0, v is picked in the cover. $c(e)$ is the amount that we charge an edge e. In particular:

1. Initialization
$C := \emptyset$
$\forall v \in V$, $t(v) := w(v)$
$\forall e \in E$, $c(e) := 0$
2. While C is not a vertex cover do:
   Pick uncovered edge, say $\{u, v\}$
   Let $m := \min\{t(u), t(v)\}$
   $t(u) := t(u) - m$
   $t(v) := t(v) - m$
   $c(u, v) := m$
   Include in C all vertices v that have $t(v) = 0$
3. Output C

Argue that this is a factor 2 approximation algorithm for weighted vertex cover.

3. Theory of Linear Inequalities

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\} \subseteq [0, 1]^n$ with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ be a polytope contained in the 0/1 cube; in particular the bound inequalities $0 \leq x \leq 1$ are valid for P.

For $i \in [n]$ we consider the following procedure:

1. Generate the nonlinear system $(b - Ax)x_i \geq 0$, $(b - Ax)(1 - x_i) \geq 0$.
2. Relinearize the system by replacing $x_j x_i$ with y_j whenever $i \neq j$ and x_j whenever $i = j$. We obtain a new, higher dimensional polyhedron M_i.

3. Define $P_i := \text{proj}_x M_i$.

Finally define $P^1 := \bigcap_{i \in [n]} P_i$. This polyhedron is a strengthening of the original formulation of P.

Prove the following:

$$P^1 = \bigcap_{i \in [n]} \text{conv}\left((P \cap \{x \mid x_i = 0\}) \cup (P \cap \{x \mid x_i = 1\}) \right)$$

4. Combinatorial Optimization

Consider the following generalization of matroids. Given a ground set E and a family $\mathcal{F}$ of subsets of E we say that $(E, \mathcal{F})$ is *k-extensible* if the following hold: (i) if $A \in \mathcal{F}$ and $B \subseteq A$, then $B \in \mathcal{F}$; (ii) consider $A, B \in \mathcal{F}$ with $A \subseteq B$; if $e \in E$ is such that $A + e \in \mathcal{F}$, then there is a set $K \subseteq B - A$ of size at most k such that $B - K + e$ belongs to $\mathcal{F}$.

Given $(E, \mathcal{F})$ k-extensible and a weight function $w : E \to \mathbb{R}$ (extended to sets as usual by $w(A) = \sum_{e \in A} w(e)$), consider the greedy algorithm for $\max_{S \in \mathcal{F}} w(S)$: (0) Start with $S = \emptyset$; (1) pick an element $e \in E - S$ with largest weight that satisfies $S + e \in \mathcal{F}$, and update $S \leftarrow S + e$ (if no such element exists, stop); (2) Repeat the previous step.

1. Let e_i be the element chosen by the greedy algorithm in step i, and let S_i be the set obtained at the end of step i (so $S_i = e_1 + \ldots + e_i$). Given any set $A \in \mathcal{F}$, let $OPT(A) = \max\{w(B) : B \supseteq A, B \in \mathcal{F}\}$ (i.e. the best extension of A in $\mathcal{F}$). Show that for all i

$$w(OPT(S_i)) \geq w(OPT(S_{i-1})) - k \cdot w(e_i).$$

2. Show that the last set S_ℓ computed by the greedy algorithm satisfies $w(S_\ell) \geq \frac{1}{k+1} w^*$, where $w^* = \max\{w(A) : A \in \mathcal{F}\}$.

3. Given a graph $G = (V, E)$ and $b \in \mathbb{R}^V_+$, a set $S \subseteq E$ is a *b-matching* if S has at most b_v edges incidents to vertex v, for all $v \in V$. Give a polytime algorithm for finding a b-matching with at least 1/3 as many edges as the largest b-matching. (No need to analyze the running-time of the algorithm.)

5. Graph Theory

Let G be a simple plane graph of minimum degree at least three. Prove that G has either a vertex of degree three incident with a face of size at most five, or a face of size three incident with a vertex of degree at most five.

6. Probabilistic methods

Let H denote the graph on five vertices consisting of a copy of K_4 along with an additional edge attached to one of the 4 vertices. (Recall that K_4 is the complete graph on 4 vertices.) Let $G_{n,p}$ denote the usual Erdős-Rényi random graph on n vertices with the edge probability $p = p(n)$. Then

(i) Show that

$$\Pr(G_{n,p} \text{ contains a copy of } H) \to 0 \text{ if } p \ll n^{-2/3};$$

(ii) Let $G_1 = G_{n,p/2}$. Show that

$$\Pr(G_1 \text{ contains a copy of } K_4) \to 1 \text{ if } p \gg n^{-2/3}.$$

(iii) Let G be the union of two independent copies of $G_{n,p/2}$. (i.e., on the same set of vertices, but the edge set is taken as the union of edge sets.) Prove that

$$\Pr(G \text{ contains a copy of } H) \to 1 \text{ if } p \gg n^{-2/3}.$$

(iv) Conclude that

$$\Pr(G_{n,p} \text{ contains a copy of } H) \to 1 \text{ if } p \gg n^{-2/3}.$$

(In the above, we are using the fairly standard notation: $p(n) \ll f(n)$ means that $p(n)/f(n) \to 0$, as $n \to \infty$, and similarly, $p(n) \gg f(n)$ means that $p(n)/f(n) \to \infty$, as $n \to \infty$.)

7. Algebra

Let A be $n \times n$ matrix with rational entries and let $p \in \mathbb{N}$ be a prime. Show that A cannot satisfy

$$A^{n+1} - pA^n = pI,$$

where I is the identity matrix.

7. Linear Algebra

Let A be a $n \times n$ matrix and v a vector in $\mathbb{R}^n$ such that the set $\{v, Av, A^2v, \ldots, A^{n-1}v\}$ is linearly independent. Show that any matrix B that commutes with A can be written as a polynomial in A.