1. Computability, Complexity and Algorithms

Given a simple directed graph $G = (V, E)$, a cycle cover is a set of vertex-disjoint directed cycles that cover all vertices of the graph.

1. Show that there is a polynomial-time algorithm to find a cycle cover of a directed graph if one exists.
2. Show that deciding if a directed graph has a cycle cover with at most k cycles, for any fixed integer $k \geq 1$, is NP-complete.
3. Show that deciding if a directed graph has a cycle cover where each cycle has at least 1% of the vertices is NP-complete.

2. Analysis of Algorithms

The following LP-relaxation is exact for the maximum weight matching problem in bipartite graphs but not in general graphs. Give a primal–dual algorithm, relaxing complementary slackness conditions appropriately, to show that the integrality gap of this LP is $\geq 1/2$. What is the best upper bound you can place on the integrality gap?

$$\begin{aligned} \text{maximize} \quad & \sum_e w_e x_e & & (1) \\ \text{subject to} \quad & \sum_{e:\, e \text{ incident at } v} x_e \leq 1, & & v \in V \\ & x_e \geq 0, & & e \in E \end{aligned}$$

3. Theory of Linear Inequalities

Let $P \subseteq [0,1]^n$ be an integral polytope contained in the 0/1 cube, i.e., the polytope as 0/1 vertices. The goal is to maximize an objective $c \in \mathbb{Z}^n$ over P. You are given a feasible integral solution $\bar{x} \in P$ and access to the polytope P is restricted to querying the following oracle:

ℓ_1-penalty oracle:
Input: $x_0 \in P$ *integral,* $\lambda \in \mathbb{R}_+$, *objective* $c \in \mathbb{Z}^n$
Output: $x \in P$ *integral with*

$$c(x - x_0) - \lambda \|x - x_0\|_1 > 0,$$

if such an x exists, otherwise return INFEASIBLE.

Consider the following simple scaling algorithm, where $C := \|c\|_\infty$.

1. Initialize $\lambda \leftarrow 2C$ and $x_0 \leftarrow \bar{x}$.
2. Repeat
   (a) Query oracle with x_0, c, λ.
   (b) IF the oracle returns a point x, then set $x_0 \leftarrow x$.
   (c) ELSE if the oracle returns INFEASIBLE, then set $\lambda \leftarrow \lambda/2$.
3. Until $\lambda < 1/n$.
4. Return x_0.

Task.

- Prove that the algorithm optimizes c over P with $O(n \log nC)$ oracle calls.
- Bonus: Can you further reduce the number of oracle calls to $O(n \log C)$, via a small modification to the algorithm?

Hint. Suppose that for a given choice $\lambda \in \mathbb{R}_+$ the oracle returns INFEASIBLE. Then in particular, also for the integral solution $x^* \in P$ that maximizes c, it holds:

$$\frac{c(x^* - x_0)}{\|x^* - x_0\|_1} \leq \lambda$$

4. Combinatorial Optimization

In the (fractional) multi-commodity flow problem, we are given a directed graph $G = (V, E)$ and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices of G, a capacity function $c : E \to \mathbb{Q}_{\geq 0}$, and demands $d_1, \ldots, d_k$, and we seek to find for each $i = 1, \ldots, k$ an $s_i - t_i$-flow $x_i \in \mathbb{Q}^E_{\geq 0}$ so that x_i has value d_i and so that for each arc e of G: $\sum_{i=1}^k x_i(e) \leq c(e)$.

Question 1. Show with Farkas' Lemma that the multicommodity flow problem has a solution if and only if for each 'length' function $l : E \to \mathbb{Q}_{\geq 0}$ one has: $\sum_{i=1}^k d_i dist_l(s_i, t_i) \leq \sum_{e \in E} l(e)c(e)$. (Here $dist_l(s, t)$ denotes the length of a shortest $s - t$ path with respect to l.)

Question 2. The cut condition states that for each $W \subseteq V$, the capacity of $\delta^{out}(W)$ is not less than the demand of W, where the capacity of $\delta^{out}(W)$ is $\text{cap}(\delta^{out}(W)) := \sum(c(e) : e \in \delta^{out}(W))$ and the demand of W is $\sum(d_i : s_i \in W \text{ and } t_i \notin W)$. Interpret the cut condition as a special case of the condition in Question 1.

5. Graph Theory

We are given two square sheets of paper, each of area 2015. Each sheet is divided into 2015 polygons of area 1 (the divisions may be different). One sheet is placed on top of the other. Show that we can place 2015 pins in such a way that the interior of each of the 4030 polygons is pierced.

6. Probabilistic methods

Consider the random graph $G := G_{n,p}$ with $p := p(n) = 1/(6\sqrt{n})$, and let S be a fixed subset of $k \geq 2$ vertices of G, where $(k/6000 \ln k)^2 \leq n$. Let Y be the maximum size of a set of edge-disjoint triangles in G such that every triangle in the set has at least two vertices in S. Prove that for every positive integer t

$$\Pr(Y \geq t) \leq \frac{(30k \ln k)^t}{t!},$$

and deduce that

$$\Pr(Y \geq 120k \ln k) < k^{-3k}.$$

You may assume that k is sufficiently large.
Remark. The constant "3" in the last expression may be improved, but to do so may require a calculator. The stated bound can be derived using mental arithmetic only.

7. Algebra

Which of the following rings are isomorphic? Justify your answers.

1. $R_0 = \mathbb{F}_5[X]/(X^2)$
2. $R_1 = \mathbb{F}_5[X]/(X^2 - 1)$
3. $R_2 = \mathbb{F}_5[X]/(X^2 - 2)$
4. $R_3 = \mathbb{F}_5[X]/(X^2 - 3)$