1. Graph Theory

Let G be a planar triangulation on $n \geq 4$ vertices, let G have no loops or parallel edges, and let $d_1, d_2, \ldots, d_n$ be the degree sequence of G. Prove that $d_1^2 + d_2^2 + \cdots + d_n^2 \leq 2n^2 + 12n - 44$, and for every integer $n \geq 4$ construct a planar triangulation on n vertices with no loops or parallel edges for which equality holds. (You may want to first prove the inequality for triangulations of minimum degree at least four, and then proceed by induction.)

Solution: We proceed by induction on n. If $n = 4$, then G is the complete graph on four vertices, and the inequality holds with equality. We may therefore assume that $n \geq 5$, and that the inquality holds for all integers smaller than n.

Assume first that $d_i \geq 4$ for all $i = 1, 2, \ldots, n$. Then $4n \leq d_1 + d_2 + \cdots + d_n = 2|E(G)| = 6n - 12$, and so $n \geq 6$. Since for $x \geq y$ we have $(x+1)^2 + (y-1)^2 \geq x^2 + y^2$, and $(n-1) + (n-3) + 4(n-2) = 6n - 12$, we deduce that

$$d_1^2 + d_2^2 + \cdots + d_n^2 \leq (n-1)^2 + (n-3)^2 + 16(n-2) \leq 2n^2 + 12n - 44,$$

as desired.

Thus we may assume that G has a vertex of degree at most three. But G has at least four vertices, and hence has minimum degree at least three. Thus G has a vertex v of degree exactly three; let v_1, v_2, v_3 be its neighbors, let d_1, d_2, d_3 be their degrees in G, and let $d_n = 3$ be the degree of v.Then $G \backslash v$ is a triangulation with degree sequence $d_1 - 1, d_2 - 1, d_3 - 1, d_4, \ldots, d_{n-1}$. Furthermore, at most one vertex of $G \backslash v$ is adjacent to all of v_1, v_2, v_3; thus $d_1 + d_2 + d_3 \leq 2n + 2$. By induction

$$d_1^2 + d_2^2 + \cdots + d_n^2 = (d_1 - 1)^2 + (d_2 - 1)^2 + (d_3 - 1)^2 + d_4^2 + \cdots + d_{n-1}^2 + 9 + 2(d_1 + d_2 + d_3) - 3 \leq$$
$$2(n-1)^2 + 12(n-1) - 44 + 9 + 4n + 4 - 3 = 2n^2 + 12n - 44,$$

as desired. Triangulations satisfying the inequality with equality can be obtained from K_4 by repeatedly inserting vertices of degree three inside a face.

2. Probability

1. The interval $[0, 1]$ is partitioned into n disjoint sub-intervals with lengths $p_1, p_2, \ldots, p_n$, and the *entropy* of this partition is defined to be

$$h = -\sum_{i=1}^{n} p_i \log p_i.$$

Let $X_1, X_2, \ldots$ be independent random variables having uniform distribution on $[0, 1]$, and let $Z_m(i)$ be the number of the $X_1, X_2, \ldots . X_m$ which lie in the i-th interval of the partition above. Define the quantity

$$R_m = \prod_{i=1}^{n} p_i^{Z_m(i)}.$$

Prove that, almost surely,

$$\lim_{m \to \infty} \frac{\log R_m}{m} = -h.$$

2. Suppose that an airplane engine will fail, when in flight, with probability $1-p$ independently from engine to engine. Suppose that the airplane will make a successful flight if at least 50% of its engines remain operative. For what values of p is a four-engine plane preferable to a two-engine plane?

Solution:
1. Let $I_{i,j}$ be the indicator function of the event that X_j lies in the i-th interval. Then

$$\log R_m = \sum_{i=1}^{n} Z_m(i) \log p_i = \sum_{i=1}^{n} \sum_{j=1}^{m} I_{i,j} \log p_i = \sum_{j=1}^{m} Y_j,$$

where, for $1 \le j \le m, Y_j = \sum_{i=1}^{n} I_{i,j} \log p_i$ is the sum of independent identically distributed random variables with mean

$$E[Y_j] = \sum_{i=1}^{n} p_i \log p_i = -h.$$

By the strong law of large numbers, $\lim_{m\to\infty}(1/m) \log R_m = -h$ almost surely.

2. A successful flight for the two engine plane means that both engines do not fail, and this occurs with probability $1-(1-p)^2$. A successful flight with a four engine plane means that three or engines do not fail, and this occurs with probability $1-[4p(1-p)^3+(1-p)^4]$. The solution requires finding the range of p such that $1-[4p(1-p)^3+(1-p)^4] > 1-(1-p)^2$. Simple algebra yields $2/3 < p < 1$.

3. Analysis of Algorithms

Let $G = (V, E)$ be an undirected graph. A subset $V' \subseteq V$ is a vertex cover for G if every edge has at least one end in V'. The k-vertex cover problem is the following: given an undirected graph G and an integer $k \le |V|$, find a vertex cover consisting of at most k vertices. The k-vertex cover problem is $\mathcal{NP}$-Hard. Using the following observation, devise an $O(|V|k + k^2 2^k)$ algorithm to find a k-vertex cover in a graph if one exists:
Observation: For each vertex v, either v or all its adjacent vertices have to be in a vertex cover. Therefore, if v is not in a vertex cover C then all its neighbors have to be in C. Let v be a vertex with degree $> k$. Suppose v is not in a k-vertex cover C. By the previous observation, all its neighbors have to be in the vertex cover C. But, they cannot all be in C since there are more than k neighbors of v. Therefore, every vertex with degree $> k$ must be part of a k-vertex cover.

Solution: Not available.

4. Combinatorial Optimization

Consider two systems of linear inequalities $Ax \leq b$ and $Cx \leq d$, and let $P = \{x : Ax \leq b\}$ and $Q = \{x : Cx \leq d\}$. Let x^* be a specified vector. Formulate, as a polynomial-sized (in the size of the two linear systems) linear-programming model, the problem of determining if there exists an inequality $w^T x \leq \delta$ that is valid for P and Q, but violated by x^*. Modify (if necessary) your LP model so that if such an inequality exists, the model finds one that maximizes the violation $w^T x^* - \delta$ among all vectors w with L_1 norm $\|w\|_1 = 1$.

Solution. A solution is available upon request.

5. Theory of Linear Inequalities

Let $G = (V, E)$ be a graph with nodes V and edges E. Edmonds' matching-polyhedron theorem implies that the linear-programming problem

$$\max \sum (w_e x_e : e \in E) \tag{1}$$

$$\sum (x_e : v \text{ is an end of } e) \leq 1 \quad \forall\, v \in V, \tag{2}$$

$$\sum (x_e : \text{both ends of } e \text{ are in } S) \leq (|S| - 1)/2 \quad \forall\, S \subseteq V, |S| \text{ odd}, \tag{3}$$

$$x_e \geq 0, \ \forall\, e \in E \tag{4}$$

has an integer optimal solution for any objective vector $w = (w_e : e \in E)$. If each component of w is integer, then it is known that there exists an optimal solution to the dual LP of (1) such that the dual variables corresponding to constraints (2) are half-integer valued and the dual variables corresponding to constraints (3) are integer valued. Use this fact to show that the linear system consisting of inequalities (2), (3), (4) is totally dual integral.

NOTE: No credit will be given for just stating that it is already known that the system (2), (3), (4) is totally dual integral.

Solution. A solution is available upon request.

6. Algebra

Prove that if G is a group having no subgroup of index 2, then any subgroup of index 3 is normal in G.

Solution: Let H be a subgroup of index 3, and consider the action of G by left multiplication on the set G/H of left cosets of H in G. Since $|G/H| = 3$, this action defines a homomorphism $\varphi : G \to S_3$. The kernel K of φ is the largest normal subgroup of G contained in H. Let G' be the image of φ. By the first isomorphism theorem, $G' \cong G/K$. Since $K \leq H$, we have $|G/K| \geq |G/H| = 3$, so G' is either A_3 or S_3. If $G' = S_3$, then φ is surjective. But then $\varphi^{-1}(A_3)$ is a subgroup of index 2 in G, contradicting our hypothesis. Thus $G' = A_3$, which means that $|G/K| = |G/H| = 3$ and thus $K = H$. It follows that H is normal in G.

7. Randomized Algorithms

For an undirected graph $G = (V, E)$, a 3-way cut is a subset S of edges whose removal from G breaks the graph into at least 3 components. The size of the 3-way cut is the number of edges in S. Use a randomized algorithm to prove that there are $O(n^4)$ 3-way cuts of minimum size.

Solution. We consider a variant of Karger's min-cut algorithm to find a min 3-way cut. Then we prove it finds any specific 3-way cut with probability $\Omega(n^{-4})$.

The randomized algorithm consists of two steps : in the first step, we run Karger min-cut algorithm till we are left with just 4 supernodes. This is the probabilistic part of the algorithm. In the second step, we go through all possible 3-way-cuts among these 4 supernodes and pick the smallest one as our candidate min-cut.

Assume that the graph has n vertices and the min-3-way-cut is of size k.

Lemma. Let u, v be two vertices of the graph. Then $d(u) + d(v) \geq k$.

Proof. Consider the 3-cut $(\{u\}, \{v\}, G - \{u, v\})$. We know that this cut has at least k edges across it.

The number of edges contributing to this 3-way cut are $d(u) + d(v) - 1$ or $d(u) + d(v)$, depending on whether there is an edge between u and v.

There are two cases. If $(u, v) \in E$, then the size of this 3-way cut is $d(u) + d(v) - 1$. We get the inequality $d(u) + d(v) - 1 \geq k$ which is $d(u) + d(v) \geq k + 1$.

If (u, v) is not an edge, then the size of this 3-way cut is $d(u)+d(v)$. In this case we get $d(u)+d(v) \geq k$. The weaker of these two is $d(u) + d(v) \geq k$. $\square$We know that $\sum_{v \in V} d(v) = 2|E|$. Now consider the sum

$$S = \sum_{i=1}^{n-1} (d(v_i) + d(v_{i+1})) + (d(v_n) + d(v_1))$$

By the above lemma, each pair has sum $\geq k$ and there are n pairs, so the total sum $S \geq kn$. Now these pairs form a cycle, so for each vertex v, $d(v)$ is counted twice. Therefore $S = 4|E|$. This gives us that $4|E| \geq kn$ or $|E| \geq kn/4$.

Now consider a fixed min 3-way cut (A, B, C) in the graph. The probability that an edge of the cut is contracted in the first interation is $\leq k/(kN/4) = 4/N$. Therefore, the probability that the specified 3-way cut is preserved is $\geq (1 - 4/N) = (N - 4)/N$. At the second iteration, this probability will be $(1 - \frac{4}{N-1}) = (N - 5)/(N - 1)$.We do this the last time for 5 vertices contracting into 4, for this case the probabilitywill be $(5 - 4)/5$. (For 4 vertices, $4/N = 1$, so we instead go through the 6 possible 3-cuts).

Multiplying these probabilities, we get the product :

$$\frac{n-4}{n} \cdot \frac{n-5}{n-1} \cdots \frac{5-4}{4} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{n(n-1)(n-2)(n-3)} > \frac{24}{n^4}.$$

Therefore, the probability the 3-way cut under consideration is preserved down to 4 vertices is $\Omega(n^{-4})$, and then there are at most 4 min 3-way cuts remaining in this graph of 4 supernodes. Therefore, the total number of 3-way cuts of minimum size is $O(n^4)$.

7. Approximation Algorithms

1. Consider TSP (Traveling Salesman Problem) in the special case of directed graphs, called $\{1, 2\}$-graphs. These are complete graphs, with each directed edge of weight 1 or 2. (Note: If u and v are vertices of a complete directed graph, then both the directed edges (u, v) and (v, u) are present. If the edges of the graph are weighted, then the weights of (u, v) and (v, u) may be different). (a) Give a polynomial time 3/2 approximation algorithm for TSP on $\{1, 2\}$-graphs. Hint: Start by giving a polynomial time algorithm that finds a minimum weight cycle cover (a set of disjoint cycles such that each vertex belongs to exactly one cycle). (b) Can you give a polynomial time algorithm with approximation guarantee 4/3?

2. Consider the following generalization of the standard metric facility location problem. G is a bipartite graph with bipartition (F, C), where F is a set of *facilities* and C is a set of *cities*. Let f_i be the cost of opening facility i and let c_{ij} be the cost of serving one unit of demand for city j by connecting it to (open) facility i. The connection costs satisfy triangular inequality. In addition, each city j has a nonnegative demand d_{ij}, and any open facility can serve this demand. The cost of serving this demand via open facility i is $c_{ij}d_j$. The problem is to find a subset $I \subseteq F$ of facilities that should be opened, and a function $\phi : C \to I$, assigning cities to open facilities in such a way that the total cost of opening facilities and serving the demand of all cities by connecting them to open facilities is minimized. The IP expressing metric facility location with general city demands is:

$$
\begin{array}{lll}
\text{minimize} & \sum_{i\in F, j\in C} c_{ij}d_jx_{ij} + \sum_{i\in F} f_iy_i & \\
\text{sunject to} & \sum_{i\in F} x_{ij} \geq 1 & j \in C \\
 & y_i - x_{ij} \geq 0 & i \in F, j \in C \\
 & x_{ij} \in \{0, 1\} & i \in F, j \in C \\
 & y_i \in \{0, 1\} & i \in F
\end{array}
$$

Give a polynomial time factor 3 approximation algorithm for the above IP. Hint: Write an LP-relaxation, and proceed along the standard primal dual factor 3 approximation algorithm for metric facility location. However, raise the dual variables a_j at rate d_j.

Solution. Not available.

7. Computational Complexity

Let $CLQ = \{G, k \mid G \text{ is a graph}, k \text{ is an integer and } G \text{ has a clique of size } k\}$.
Let $\mathcal{S} = \{C \mid C \text{ is a circuit with } m \text{ inputs that accepts all length } m \text{ encodings of elements of } CLQ\}$.
1. Show that $\mathcal{S}$ is in $\mathcal{CONP}$.
2. Prove: If for all integers $m > 0$, there exists a circuit C_m with m inputs and size $O(m^k)$, for some constant k, such that C_m accepts all length m encodings of elements of CLQ, then $\Pi_2^P \subseteq \Sigma_2^P$.

Solution. Solution is available upon request.