## 1. Theory of Linear Inequalities

Let $a_0, a_1, \ldots a_n$ be integral vectors in $\mathbf{R}^d$ and let $A$ be the matrix having $a_0, a_1, \ldots, a_n$ as columns. For a nonnegative integer $\lambda$, call $\{a_0, a_1, \ldots, a_n\}$ a $(a_0, \lambda)$-*Hilbert basis* if every integral vector $b$ in cone$(\{a_0, a_1, \ldots, a_n\})$ can be written as integral combination

$$b = \sum(\gamma_i a_i : i = 0, 1, \ldots n)$$

where $\gamma_0 + \lambda \geq 0$ and $\gamma_i \geq 0$ for $i = 1, 2, \ldots, n$. Suppose cone$\{a_0, a_1, \ldots, a_n\}$ is a pointed cone. Show that $\{a_0, a_1, \ldots, a_n\}$ is **not** a $(a_0, \lambda)$-Hilbert basis if and only if there is an integral vector $b$ in cone$(\{a_0, a_1, \ldots, a_n\})$ such that $b + \lambda a_0$ is not a nonnegative integer combination of $a_0, a_1, \ldots, a_n$ and

$$max\{\mathbf{1}^T x | Ax = b, x \geq 0\} < d.$$

## 2. Combinatorial Optimization

Let $G = (V, E)$ be a graph and let $k$ be the cardinality of a maximum matching in $G$. Let $E_1$ and $E_2$ be non-empty subsets of $E$ with $E_1 \cup E_2 = E$. For $i = 1, 2$, let $k_i$ be the cardinality of a maximum matching of $G$ contained in $E_i$. If $k_1 + k_2 = k$, then $(E_1, E_2)$ is called a *matching separation* of $G$. If $G$ has no matching separation then it is called *matching nonseparable*.

A graph $G = (V, E)$ is called *factor-critical* if for each vertex $v \in V$ the graph obtained by deleting $v$ from $G$ has a perfect matching.

Show that a graph $G$ with no isolated vertices is matching nonseparable if and only if $G$ is isomorphic to $K_{1,t}$ for some $t$ **or** $G$ is factor-critical and 2-vertex connected.

## 3. Analysis of Algorithms

**(a)** Given $n$ points in the plane, design an $O(n^2 logn)$ algorithm that determines if any three points are colinear.

**(b)** Assume you have $n$ square containers of different sizes with matching lids. Consider the following randomized algorithm for finding the largest container or lid.

Draw a lid uniformly at random from the set of $n$ lids, and draw a container uniformly at random from the set of $n$ containers. If the lid is smaller than the container, discard the lid, draw a new lid uniformly at random from the remaining lids, and repeat. If the container is smaller than the lid discard the container, draw a new container uniformly at random from the remaining containers, and repeat. If they are the same size flip a fair coin to decide which one to discard and repeat. Stop when you run out of containers or lids. When you stop, the container (or lid) you are holding is the largest.

What is the expected number of lid-container tests performed by this algorithm? Prove your answer is correct.

## 4. Approximation Algorithms

**1.** Consider the $(2 - 2/k)$ factor algorithm for the multiway cut problem that operates by finding minimum cuts separating each terminal from all the rest. Show that the analogous algorithm for the node multiway cut problem, based on isolating cuts, does not achieve a constant factor. What is the best factor you can prove for this algorithm?
**2.** The multiway cut problem also possesses the half-integrality property. Give a suitable LP for the multiway cut problem and prove this fact.

## 5. Graph Theory

For a graph $G$ we use $e(G)$ to denote the number of edges of $G$. Let $H$ be a spanning subgraph of a graph $G$ such that every component of $H$ is an induced subgraph of $G$ and is bipartite. Prove that $G$ has a bipartite subgraph with at least $e(G)/2 + e(H)/2$ edges.

## 6. Probability

Suppose $(X_n)_{n=1}^{\infty}$ is a sequence of independent standard Gaussian random variables. Prove:

$$\mathsf{P}\left\{\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\ln n}} = 1\right\} = 1.$$

**Hint.** Use the asymptotic relation

$$\frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2} dy \ \sim \ \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}, \quad x \to \infty, \tag{1}$$

to show

$$\mathsf{P}\left\{\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\ln n}} \le 1\right\} = 1, \tag{2}$$

and

$$\mathsf{P}\left\{\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\ln n}} \ge 1\right\} = 1. \tag{3}$$

## 7. Algebra

Let $a_n$ denote the Fibonacci sequence $a_0 = 0$, $a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$, and let $b_n = (a_n)^2$. Prove that $b_n$ satisfies a linear recursion relation.