1. Computability, Complexity and Algorithms

Let $G = (V, E)$ be a directed graph with integer weights w_{ij} on its edges. The vertex set V has been partitioned into m subsets $V_1, \ldots, V_m$, each containing k vertices. Each vertex $u \in V_r$, for $1 \leq r \leq m-1$, has edges going to every vertex $v \in V_{r+1}$. These are all the edges of the graph.

1. Given two vertices $a \in V_1$ and $b \in V_m$ and an integer W, give an algorithm to determine whether there is a directed path from a to b of total weight at least W. The time complexity of your algorithm should be polynomial in m and k.
2. Give a nondeterministic algorithm for the same problem that uses only $O(\log m + \log k)$ space.
3. Give a deterministic algorithm that uses $O((\log m + \log k)^2)$ space.

2. Analysis of Algorithms

The 1-2 directed TSP is the following problem: Given a complete directed graph G on n vertices and with weights w on its edges such that all $w(e) \in \{1, 2\}$, find a Hamilton cycle of minimum weight. Give a polynomial time 3/2 factor approximation algorithm for the 1-2 directed TSP. (Hint: Cover the vertices with cycles of minimum total weight, then patch the cycles.)

3. Theory of Linear Inequalities

Recall that given a polyhedron $P \subseteq \mathbb{R}^n$, $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$, we have that

$$P_I \subseteq P^{\pi,\pi_0} := \operatorname{conv}\{(P \cap \{x \in \mathbb{R}^n \mid \pi x \leq \pi_0\}) \cup (P \cap \{x \in \mathbb{R}^n \mid \pi x \geq \pi_0 + 1\})\},$$

where P_I is the integer hull of P. The split closure $S(P)$ is defined as $S(P) := \bigcap_{\pi \in \mathbb{Z}^n, \pi_0 \in \mathbb{Z}} P^{\pi,\pi_0}$.

Suppose that $A \in \mathbb{Z}^{m \times n}$ has the property that on removing any column of A, the remaining matrix is totally unimodular. Prove that if $P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$ and $b \in \mathbb{Z}^m$, then $S(P) = P_I$.

4. Combinatorial Optimization

Observe that the Petersen graph has two perfect matchings, $\mathcal{M}_1$ and $\mathcal{M}_2$, such that $\mathcal{M}_1$ and $\mathcal{M}_2$ intersect in exactly one edge. Equivalently, the matching $\mathcal{M}_2$ contains $\frac{2}{5}$ of the edges of $E(G) \setminus \mathcal{M}_1$.

Let G be 3-regular graph with the property that for all $U \subseteq V(G)$ with $|U| \geq 2$, $|V(G) \setminus U| \geq 2$, we have that at least four edges of G have one end in U and the other end in $V(G) \setminus U$. Prove that for all perfect matchings $\mathcal{M}$ of G, there exists a matching $\mathcal{M}'$ such that $|\mathcal{M}' \cap (E(G) \setminus \mathcal{M})| \geq \frac{2}{5}|E(G) \setminus \mathcal{M}|$.

5. Graph Theory

Let G be a simple 3-regular graph, and let k be its edge-chromatic number. Prove that if every two k-edge-colorings of G differ by a permutation of colors, then $k = 3$ and G has three distinct Hamiltonian cycles.

6. Probabilistic methods

Show that there exists an absolute constant c so that if $\{S_i : 1 \leq i \leq n\}$ is any sequence of sets with $|S_i| \geq c$, for all $i = 1, 2, \ldots, n$, then there exists a sequence $\{x_i : 1 \leq i \leq n\}$ with $x_i \in S_i$, for all $i = 1, 2, \ldots, n$, which is square-free, i.e., there is no pair i, j with $1 \leq i < j \leq 2j - i - 1 \leq n$ so that $x_{i+k} = x_{j+k}$ for all $k = 0, 1, \ldots, j - i - 1$. Hint: This is an application of the asymmetric version of the Lovasz Local Lemma.

7. Algebra

Let n be an integer > 1 and let S_n be the symmetric group on the set $\{1, \ldots, n\}$. Let $G \subset S_n$ be a subgroup which acts transitively on $\{1, \ldots, n\}$. Prove that there is an element $\sigma \in G$ such that $\sigma(a) \neq a$ for all $a \in \{1, \ldots, n\}$. Give a complete proof using only basic facts about groups acting on sets.