## 1. Computability, Complexity and Algorithms

Consider two sets $A$ and $B$, each having $n$ integers in the range from 0 to $8n$ where $n$ is a power of 2. We wish to compute the *Cartesian sum* of $A$ and $B$, defined by:

$$C = \{x + y : x \in A \text{ and } y \in B\}.$$

We want to find the set of elements in $C$ and also the number of times each element of $C$ is realized as a sum of elements in $A$ and $B$.

**Part (a):** Give an algorithm to compute the Cartesian sum $C$ **by a reduction to FFT**. State the running time (as fast as possible in $O()$ notation).

**Part (b):** Extend your algorithm to obtain the number of times each $i \in C$ is realized as a sum of elements in $A$ and $B$.

*Example:* for $A = [1, 2, 3]$ and $B = [2, 3]$ then $C = [3, 4, 5, 6]$ and the solution to the Cartesian Sum problem is:

3 appears and is obtainable in 1 way,
4 appears and is obtainable in 2 ways,
5 appears and is obtainable in 2 ways,
6 appears and is obtainable in 1 way.

## 2. Analysis of Algorithms

Recall that computing the number of perfect matchings in a graph $G = (V, E)$ is #P-complete. For this problem assume that you are given an oracle that returns the number of perfect matchings in a given graph in one time step.

(i) A graph is said to be *matching covered* if every edge of it participates in some perfect matching. Given graph $G = (V, E)$ show how to obtain, in polynomial time, a subgraph $G' = (V, E')$, with $E' \subseteq E$ such that G' is matching covered and the number of perfect matchings in $G$ and $G'$ is the same.

Recall that the perfect matching polytope for a bipartite graph $G = (V, E)$ is defined in $\mathbb{R}^E$ and is given by the following set of linear equalities and inequalities.

$$\begin{aligned} x(\delta(v)) &= 1 \quad \forall v \in V, \\ x_e &\geq 0 \quad \forall e \in E. \end{aligned} \tag{1}$$

The equation says that the total $x$ value of edges incident at each vertex $v$ is 1.

(ii) Give a polynomial time algorithm for finding a point in the interior of the perfect matching polytope for a connected, matching covered bipartite graph $G = (V, E)$.

## 3. Theory of Linear Inequalities

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\} \subseteq [0,1]^n$ with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ be a polytope contained in the 0/1 cube; in particular the bound inequalities $0 \leq x \leq 1$ are valid for $P$.

For $i \in [n]$ we consider the following procedure:

1. Generate the nonlinear system $(b - Ax)x_i \geq 0$, $(b - Ax)(1 - x_i) \geq 0$.
2. Relinearize the system by replacing $x_j x_i$ with $y_j$ whenever $i \neq j$ and $x_j$ whenever $i = j$. We obtain a new, higher dimensional polyhedron $M_i$.
3. Define $P_i := \text{proj}_x M_i$.

Finally define $P^1 := \bigcap_{i \in [n]} P_i$. This polyhedron is a strengthening of the original formulation of $P$.

Prove the following:

$$\text{conv}(P \cap \{0,1\}^n) \subseteq P^1 \subseteq P.$$

## 4. Combinatorial Optimization

Given an integer $n$, let $\mathcal{M}_k = (U, \mathcal{I}_k)$ be a matroid for each $1 \leq k \leq n$ with $\mathcal{M}_k^* = (U, \mathcal{I}_k^*)$ its dual matroid. Consider the matroid $\mathcal{N} = (U, \mathcal{I})$ defined as $\mathcal{N} := (\mathcal{M}_1^* \vee \ldots \vee \mathcal{M}_n^*)^*$ ,i.e., it is the dual of the union of matroids $\mathcal{M}_1^*, \ldots \mathcal{M}_n^*$.

1. (4 points) Show that
$$\mathcal{I} \subseteq \bigcap_{k \in \{1,\ldots,n\}} \mathcal{I}_k.$$
2. (4 points) Let $(P_1, \ldots, P_n)$ denote a partition of $U$, i.e. $\cup_{k=1}^n P_k = U$ and each element of $U$ appears in exactly one $P_k$. Let $b_1, \ldots, b_n$ be positive integers such that $|P_k| \geq b_k$ for each $1 \leq k \leq n$. For every $1 \leq k \leq n$, consider the matroid $\mathcal{M}_k = (U, \mathcal{I}_k)$ where some subset $S$ of $U$ is in $\mathcal{I}_k$ if $|S \cap P_k| \leq b_k$ (observe that there is no restriction on elements not in $P_k$). Show that the matroid $\mathcal{N}$ as defined above is a partition matroid in the this case. Moreover, show that equality holds in the above containment.
3. (2 points) Give an example where equality does not hold in the containment in (a).

## 5. Graph Theory

Consider the graphs $G$ in which every induced subgraph $H$ has the property that the vertex-set of every maximal complete subgraph of $H$ intersects every maximal independent set in $H$.

1. Prove that every such graph $G$ is perfect.
2. Prove that these graphs $G$ are precisely the graphs with no induced subgraph isomorphic to the path on four vertices.

## 6. Probabilistic methods

Let $B_{n,n,p}$ denote the random *bipartite* graph with $n$ vertices in each part, where an edge connecting two vertices in different parts is included independently with probability $p$ (and there are no edges connecting vertices in the same part). Let $X$ be the random variable which counts the number of 4-cycles in $B_{n,n,p}$. Use Janson's inequality (or extended Janson's inequality) to prove bounds of the form

$$\Pr[X = 0] \leq \mathrm{e}^{-\Omega(n^x p^y)}$$

(a) if $0 < p < 1$ is a constant.

(b) if $0 < p = p(n) < 1$ is a function of $n$
*(Hint: you might want to distinguish different ranges of $p$, e.g., where $n^{-\alpha} \ll p \ll n^{-\beta}$ holds for suitable $\alpha, \beta > 0$)*

## 7. Algebra

Let $p$ be a prime and $\mathbb{F}_q$ be a field with $p^d$ elements. Let $f : \mathbb{F}_q \to \mathbb{F}_q$ be the map $f(x) = x^p$ for all $x$ in $\mathbb{F}_q$. Show that there exists an element $x$ in $\mathbb{F}_q$ such that $\{x, fx, \ldots, f^{d-1}x\}$ is a basis for $\mathbb{F}_q$ as an $\mathbb{F}_p$-vector space.

## 7. Linear Algebra

**Notation**. For a matrix $A \in \mathbb{R}^{n \times n}$, we write $A \geq 0$ to mean that all the entries of $A$ are nonnegative numbers.

Consider a matrix $A \in \mathbb{R}^{n \times n}$ satisfying these conditions (this is called an *$M$-matrix*):

(i) for all $i, j = 1, \ldots, n$, and $i \neq j$, $a_{ij} \leq 0$;

(ii) we can write $A = sI - B$, where $B \geq 0$, and $s \geq \rho(B)$.

Further, $A$ is an invertible $M$-matrix if, in part (ii), $s > \rho(B)$. Prove that
*$A$ is an invertible $M$-matrix if and only if $A^{-1} \geq 0$.*