1. Computability, Complexity and Algorithms

(a): Count $s - t$ Paths in DAGs: Let $G(V, E)$ be a directed acyclic graph given in adjacency list representation, and let $s \in V$ and $t \in V$ be distinct vertices. Give an $O(|V| + |E|)$ algorithm that computes the number of distinct paths from s to t in G.
(b): Count $s - t$ Paths in General Directed Graphs: Let $G(V, E)$ be a general directed graph given in adjacency matrix representation, and let $s \in V$ and $t \in V$ be distinct vertices. Argue that, if there is a polynomial-time algorithm that computes the number of distinct paths from s to t in G, then there is a polynomial-time algorithm that decides Hamiltonicity in general directed graphs.

2. Analysis of Algorithms

Matrix Identity Testing

- Recall the Schwartz-Zippel lemma:

  Lemma 1 (Schwartz-Zippel Lemma) *Let $p(x_1, \ldots, x_n)$ be a nonzero polynomial of n variables with degree d. Let S be a finite subset of $\mathbb{R}$, with at least d elements in it. If we assign $x_1, \ldots, x_n$ values from S independently and uniformly at random, then*

  $$\mathbb{P}[p(x_1, \ldots, x_n) = 0] \leq \frac{d}{|S|}.$$

  Using the aforementioned lemma, design a randomized algorithm to test whether $AB = C$, where A, B, C are three $n \times n$ matrices. Analyze the probability with which it will succeed, and analyze its runtime.

- Explain how to "boost" the above algorithm to succeed with probability $1 - \delta$.

3. Theory of Linear Inequalities

Let $e^k \in \mathbb{R}^n$ for $k = 0, \ldots, n-1$ denote the vector with the first k entries being 1 and the following $n - k$ entries being -1. Let $S = \{e^0, e^1, \ldots, e^{n-1}, -e^0, \ldots, -e^{n-1}\}$, i.e., S consists of all vectors consisting of $+1$ followed by -1 or vice versa.

1. Consider any vector $a \in \{-1, 0, 1\}^n$ such that
   (a) $\sum_{i=1}^{n} a_i = 1$, and
   (b) for all $j = 1, \ldots, n-1$, we have $0 \leq \sum_{i=1}^{j} a_i \leq 1$.

Show that $\sum_{i=1}^{n} a_i x_i \leq 1$ and $\sum_{i=1}^{n} a_i x_i \geq -1$ are valid inequalities for $conv(S)$.

2. Show that any such inequality defines a facet of $conv(S)$.

4. Combinatorial Optimization

Assume n is odd, and $G = (V, E)$ is a graph with $|V| = n$, $|E| = 2n - 2$, such that G is the union of two edge-disjoint spanning trees. Assume furthermore that half of the edges are colored red, the other half blue (where the coloring of edges is unrelated to the spanning trees). Show that G contains a spanning tree where exactly half of the edges are red and half of them blue.

5. Graph Theory

Let d be a positive integer and let G be a graph with average degree at least $8d$. Show that G contains a d-connected subgraph whose edges can be oriented so that the resulting digraph has no directed path on three vertices.

6. Probabilistic methods

Let S_n be a random string of length n, where each character is, independently, chosen uniformly at random from the alphabet $\mathcal{A} := \{A, \ldots, Z\}$. For each n, let $H_n \in \mathcal{A}^m$ be a given string of length $m = m(n) \geq 0$. We say that S_n *contains* H_n if S_n it contains a consecutive substring of length m which equals H_n. Find a threshold function $m^* = m^*(n)$ such that

$$\Pr(S_n \text{ contains } H_n) \to \begin{cases} 1 & m = o(m^*), \\ 0 & m = \omega(m^*). \end{cases}$$

7. Algebra

Let p be a prime number. Show that if G is a finite p-group, and if $N \trianglelefteq G$ is a normal subgroup of order p, then N is contained in the center of G.

7. Linear Algebra

Let A be a bistochastic matrix, that is a real $n \times n$ matrix such that

$$A_{i,j} \geq 0 \quad \forall i, j \qquad \sum_{i=1}^{n} A_{i,j} = 1 \quad \forall j \qquad \sum_{j=1}^{n} A_{i,j} = 1 \quad \forall i \ .$$

Let $a = \min_{i,j} A_{i,j}$ and let $v \in \mathbb{R}^n$ be such that $\sum_{i=1}^n v_i = 0$.
(a) Show that

$$\|Av\|_1 \leq (1 - na)\|v\|_1,$$

where $\|v\|_1 = \sum_{i=1}^n |v_i|$. Is the estimate sharp? That is, can you find A and v as above such that

$$\|Av\|_1 = (1 - na)\|v\|_1 \ ?$$

(b) Show that

$$\|Av\|_\infty \leq (1 - na)\|v\|_\infty,$$

where $\|v\|_\infty = \max_i |v_i|$. Is the estimate sharp? That is, can you find A and v as above such that

$$\|Av\|_\infty = (1 - na)\|v\|_\infty \ ?$$