1. Computability, Complexity and Algorithms

Given a simple, undirected graph $G = (V, E)$ with n vertices and an integer k, the (k, n)-CLIQUE problem is to determine whether G contains a clique of size k. The (k, n)-CLIQUE problem is NP-complete.

1. For any integer $\ell \geq 2$, show that the problem of determining whether a graph of size ℓn has a clique of size n is NP-complete, i.e., the $(n, \ell n)$-CLIQUE problem is NP-complete.

Given a graph $G = (V, E)$, and integers $k, t \geq 0$, the (k, t)-DENSE-SUBRAPH problem is to determine whether G contains a subgraph with k vertices and at least t edges.

2. Show that there is a function $e(k) = \Theta(k^{3/2})$ such that the $(k, e(k))$-DENSE-SUBGRAPH problem is NP-complete. [Hint: consider the disjoint union of a graph and one or more complete graphs.]

2. Theory of Linear Inequalities

Consider the problem:

$$\begin{array}{ll} \max & \sum_{1 \leq i < j \leq n} c_{ij} x_i x_j - \sum_{i=1}^{n} d_i x_i \\ \text{s.t.} & x \in \{0, 1\}^n \end{array}$$

Assuming c is non-negative, show that the above problem can be solved in polynomial-time.

2. Analysis of Algorithms

Consider a tree with n vertices, one of which, s, is special, but hidden from the algorithm. One can repeatedly pick a vertex u, and ask whether $u = s$ or for the first edge on the shortest path from u to s. Give an algorithm that finds s in time $O(n \log n)$ using $O(\log n)$ queries.

3. Graph Theory

Let k be a positive integer and G be a $(k+1)$-color-critical graph, i.e., $\chi(G) = k+1$ and $\chi(H) \leq k$ for any proper subgraph H of G. Show that G is k-edge-connected.

4. Algebra

Compute the degree of the splitting field of $x^{90} - 1$ over the following fields.

1. $\mathbb{F}_2$
2. $\mathbb{F}_3$
3. $\mathbb{F}_5$
4. $\mathbb{F}_7$

4. Linear Algebra

Let V be an n dimensional inner product space, A and B are linear transformations on V. Suppose A and B are selfadjoint (or Hermitian, that is $A = A^*$ and $B = B^*$) and $AB = BA$. Show that there exists an orthonormal basis of V such that with respect to this basis, both A and B are diagonal.

4. Combinatorial Optimization

1. (4 points) Let $G = (V, E)$ be a graph and let $S \subseteq V$. Let

   $$\mathcal{I} = \{A \subseteq S : A \text{ can be covered by a matching in } G\}.$$

   Show $\mathcal{M} = (S, \mathcal{I})$ is a matroid.

2. (6 points) Give a polynomial time algorithm that given a graph $G = (V, E)$ and disjoint sets $S, T \subset V$ and non-negative integers s and t, decides whether there is a matching that covers at least s vertices from S and at least t vertices from T.