1. Computability, Complexity and Algorithms

Given a simple, undirected graph $G = (V, E)$ with n vertices and an integer k, the (k, n)-CLIQUE problem is to determine whether G contains a clique of size k. The (k, n)-CLIQUE problem is NP-complete.

1. For any integer $\ell \geq 2$, show that the problem of determining whether a graph of size ℓn has a clique of size n is NP-complete, i.e., the $(n, \ell n)$-CLIQUE problem is NP-complete.

Given a graph $G = (V, E)$, and integers $k, t \geq 0$, the (k, t)-DENSE-SUBRAPH problem is to determine whether G contains a subgraph with k vertices and at least t edges.

2. Show that there is a function $e(k) = \Theta(k^{3/2})$ such that the $(k, e(k))$-DENSE-SUBGRAPH problem is NP-complete. [Hint: consider the disjoint union of a graph and one or more complete graphs.]

Solution: For (1), we reduce from the clique problem: given a graph $G = (V, E)$ on n vertices and integer $k \geq 0$, we construct a graph H with a copy of G and a clique on $n - k$ vertices, with every vertex of the $n - k$ clique connected to all vertices of G as a complete bipartite graph. In addition, create $\ell n - n - (n - k)$ isolated vertices. So H has ℓn vertices in total, and if G has a clique of size k, this induces a clique of size n in H. If H has a clique of size n, at least k of these vertices must be in G and hence G contains a clique of size k.

For (2), we reduce from the problem shown to be NP-complete in (1), with $\ell = 2$. Given an instance G of (1), i.e., a graph on $2n$ vertices, we create a new graph H with a copy of G and n disjoint cliques of size $2n+1$. Then the size of the subgraph used will be $k = n.(2n+1) + n$ and the number of edges in it will be $e(k) = n.(2n+1)(2n)/2 + n(n-1)/2 = \Theta(n^3) = \Theta(k^{3/2})$.

If H has a subgraph of size k with $e(k)$ edges, then it must contain all n $2n+1$-cliques and only n vertices from G. To see this, if the dense subgraph had more than n vertices from G, note that we can replace them with the remaining (unpicked) vertices from the disjoint cliques, and this only increases the number of edges of the subgraph, which cannot increase the number $e(k)$ above. Thus, the answer this instance of DENSE-SUBGRAPH is YES iff the original $(n, 2n)$-CLIQUE instance has a YES answer. [Note: the same construction with more cliques can be used to show hardness for $e(k) = \Theta(k^{1+\epsilon})$.]

2. Theory of Linear Inequalities

Consider the problem:

$$\begin{array}{ll} \max & \displaystyle\sum_{1 \leq i < j \leq n} c_{ij} x_i x_j - \sum_{i=1}^{n} d_i x_i \\ \text{s.t.} & x \in \{0, 1\}^n \end{array}$$

Assuming c is non-negative, show that the above problem can be solved in polynomial-time.

Solution. Consider the following equivalent integer program:

$$
\begin{aligned}
\max \quad & c^\top w - d^\top x \\
\text{s.t.} \quad & w_{ij} \le x_i \ \forall i, j \in [n] \\
& w_{ij} \le x_j \ \forall i, j \in [n] \\
& w_{ij} \ge x_1 + x_j - 1 \ \forall i, j \in [n] && (1) \\
& w_{ij} \ge 0 \ \forall i, j \in [n] && (2) \\
& 0 \le x_i \le 1 \ \forall i \in [n] \\
& x \in \mathbb{Z}^n, \ w \in \mathbb{Z}^{\frac{(n-1)n}{2}}
\end{aligned}
$$

Since $c_{ij} \ge 0$ for all $i, j \in [n]$, we do not need to lower bound w_{ij}. Thus we are able to drop constraints (**??**) and (**??**) to obtain IP model with the same optimal objective function value:

$$
\begin{aligned}
\max \quad & c^\top w - d^\top x \\
\text{s.t.} \quad & w_{ij} - x_i \le 0 \ \forall i, j \in [n] && (3) \\
& w_{ij} - x_j \le 0 \ \forall i, j \in [n] && (4) \\
& -x_i \le 0 \ \forall i \in [n] && (5) \\
& x_i \le 1 \ \forall i \in [n] && (6) \\
& x \in \mathbb{Z}^n, \ w \in \mathbb{Z}^{\frac{(n-1)n}{2}}
\end{aligned}
$$

We will show that the constraint matrix of the above system, i.e. (**??**) - (**??**), is totally unimodular (TU) and since the right-hand-side is integral, we obtain that the polyhedron defined by (**??**) - (**??**) is integral. Therefore the above problem can be solved using a linear program, i.e., in polynomial time.

By Theorem 19.3 (iv), we have to show that for any subset of columns J, there exists a partition into J^1 and J^2, so that the sum of the columns over J^1 minus the sum of the columns in J^2 is a vector of $\{-1, 0, 1\}$. Indeed, for the above system, set $J^1 = J$ and $J^2 = \emptyset$. Then it is straighforward to check that for this partition the condition is satisfied. Thus the constraint matrix is TU.

2. Analysis of Algorithms

Consider a tree with n vertices, one of which, s, is special, but hidden from the algorithm. One can repeatedly pick a vertex u, and ask whether $u = s$ or for the first edge on the shortest path from u to s. Give an algorithm that finds s in time $O(n \log n)$ using $O(\log n)$ queries.

Solution: We prove the following fact: every tree on n vertices has a vertex (known as a separator vertex) whose removal partitions this tree into components of size at most $n/2$.

Given this fact, we can proceed by repeatedly querying at such separator vertices: each such step narrows down the number of potential locations for s by a constant factor.

To show the existence of this separator vertex, suppose for a contradiction that it does not exist for some tree T on n vertices. Thus for every vertex u there exists an incident edge $f(u)$ such that the component of $T \backslash f(u)$ not containing u has strictly more than $n/2$ vertices. Since T has n vertices, it has $n-1$ edges, and hence there exist distinct vertices u, v such that $f(u) = f(v)$. It follows that both components of $T \backslash f(u)$ have strictly more than $n/2$ vertices, a contradiction.

3. Graph Theory

Let k be a positive integer and G be a $(k+1)$-color-critical graph, i.e., $\chi(G) = k+1$ and $\chi(H) \leq k$ for any proper subgraph H of G. Show that G is k-edge-connected.

Solution: Suppose G is not k-edge-connected, and let X, Y be a partition of the vertex set of the graph G such that $e(X, Y) < k$.

Since G is $(k+1)$-color-critical, $\chi(G[X]) \leq k$ and $\chi(G[Y]) \leq k$. Thus, X has a partition into independent sets $X_1, \ldots, X_k$ and Y has a partition into independent sets $Y_1, \ldots, Y_k$. (Here we allow X_i and Y_j to be empty.)

We derive a contradiction by showing that there is a permutation $(i_1 \ldots i_k)$ of $[k] = \{1, \ldots, k\}$ such that $X_j \cup Y_{i_j}$, $j \in [k]$, are independent sets in G.

Build an auxiliary bipartite graph H with partition sets $\{x_i : i \in [k]\}$ and $\{y_j : j \in [k]\}$ such that $x_i y_j \in E(H)$ iff $X_i \cup Y_j$ is an independent set in G. Since $e(X, Y) \leq k-1$, $e(H) \geq k^2 - (k-1)$. So one needs to use at least k vertices to cover all edges of H. By König's theorem, H has a perfect matching. Hence there is a permutation $(i_1 \ldots i_k)$ of $[k]$ such that $X_j \cup Y_{i_j}$, $j \in [k]$, are independent sets in G.

4. Algebra

Compute the degree of the splitting field of $x^{90} - 1$ over the following fields.

1. $\mathbb{F}_2$
2. $\mathbb{F}_3$
3. $\mathbb{F}_5$
4. $\mathbb{F}_7$

Solution:

1. $\mathbb{F}_2$: $x^{90} - 1 = (x^{45} - 1)^2$, so it is equivalent to find the splitting field of $x^{45} - 1$. The degree of this splitting field is the smallest d such that $2^d \equiv 1 \mod 45$. This is the same as finding the order of 2 in $(\mathbb{Z}/9)^* \times (\mathbb{Z}/5)^*$, which is the least common multiple of the order of 2 in $(\mathbb{Z}/9)^*$ and $(\mathbb{Z}/5)^*$. This is the LCM of 6 and 4, which gives $d = 12$.

2. $\mathbb{F}_3$: $x^{90}-1=(x^{10}-1)^9$, so it is equivalent to find the splitting field of $x^{10}-1$. The degree of this splitting field is the smallest d such that $3^d \equiv 1 \mod 10$, so $d=4$.

3. $\mathbb{F}_5$: $x^{90}-1=(x^{18}-1)^5$, so it is equivalent to find the splitting field of $x^{18}-1$. The degree of this splitting field is the smallest d such that $5^d \equiv 1 \mod 18$, which is the order of 5 in $(\mathbb{Z}/9)^* \times (\mathbb{Z}/2)^*$, which is $d=6$.

4. $\mathbb{F}_7$: The degree of this splitting field is the smallest d such that $7^d \equiv 1 \mod 90$, so we must find the order of 7 in $(\mathbb{Z}/9)^* \times (\mathbb{Z}/5)^* \times (\mathbb{Z}/2)^*$. This is the LCM of 3 (note $7 \equiv -2 \mod 9$), 4, and 1, so $d=12$.

4. Linear Algebra

Let V be an n dimensional inner product space, A and B are linear transformations on V. Suppose A and B are selfadjoint (or Hermitian, that is $A=A^*$ and $B=B^*$) and $AB=BA$. Show that there exists an orthonormal basis of V such that with respect to this basis, both A and B are diagonal.

Solution: Since A is selfadjoint, so there exists an orthonormal basis $\{v_1, v_2, \dots, v_n\}$ such that $Av_i=\lambda_i v_i$. Suppose A has p distinct eigenvalues, labeled as $\{\mu_1, \mu_2, \dots, \mu_p\}$. Let $E_{\mu_j}(A) = ker(A-\mu_j I)$ the eigenspace of A corresponding to the eigenvalue μ_j. Since A is selfadjoint, then $V=E_{\mu_1}(A) \oplus E_{\mu_2}(A) \oplus \cdots \oplus E_{\mu_p}(A)$. If $x \in E_{\mu_j}(A)$ for some j, then $ABx=BAx=B(\mu_j x)=\mu_j Bx$, so $Bx \in E_{\mu_j}(A)$. Thus $E_{\mu_j}(A)$ is invariant for B. Let $B_j = B|E_{\mu_j}(A)$, the restriction of B on $E_{\mu_j}(A)$. Then for any $x, y \in E_{\mu_j}(A)$, $\langle B_j x, y\rangle = \langle Bx, y\rangle = \langle x, B^* y\rangle = \langle x, By\rangle = \langle x, B_j y\rangle = \langle B_j^* x, y\rangle$, so $B_j = B_j^*$. Thus B_j is selfadjoint and there is an orthonormal basis $\{u_1^{(j)}, \dots, u_{m_j}^{(j)}\}$ where $m_j = \dim E_{\mu_j}(A)$ such that each $u_i^{(j)}$ is an eigenvector for B_j, for $1 \le i \le m_j$. Since each $u_i^{(j)}$ is in $E_{\mu_j}(A)$, each $u_i^{(j)}$ is an eigenvector for both A and B, and $\{u_i^{(j)} : 1 \le i \le m_j,\ 1 \le j \le p\}$ is an orthonormal basis for V. With respect to this basis, both A and B are diagonal.

4. Combinatorial Optimization

1. (4 points) Let $G=(V,E)$ be a graph and let $S \subseteq V$. Let

$$\mathcal{I} = \{A \subseteq S : A \text{ can be covered by a matching in } G\}.$$

Show $\mathcal{M}=(S,\mathcal{I})$ is a matroid.

2. (6 points) Give a polynomial time algorithm that given a graph $G=(V,E)$ and disjoint sets $S, T \subset V$ and non-negative integers s and t, decides whether there is a matching that covers at least s vertices from S and at least t vertices from T.

Solution.

1. Observe that $\mathcal{M}$ is just the restriction of the matching matroid to S and thus is a matroid. Alternatively, we verify the matroid axioms. Let $A \subseteq S$ be set covered by a matching M. For every $B \subseteq A$, clearly M also covers B and thus $B \in \mathcal{I}$. Now, let A, B be two sets in $\mathcal{I}$ such that $|A| = |B| + 1$. Let M_A be a matching covering A and let M_B be a matching covering B. If M_B covers a vertex of $A \setminus B$, say v, then $B \cup \{v\} \in \mathcal{I}$ as required. Consider $M_A \Delta M_B$ which is a collection of $M_A - M_B$ alternating paths. Observe that each vertex of $A \setminus B$ is an endpoint of such a path since it is covered by A and not covered by B. But since $|B \setminus A| < |A \setminus B|$ there must be one such path P whose other endpoint is not in $B \setminus A$. If the other endpoint is covered by M_B, it is also not in A since every such vertex is covered by M_A. Thus $M_B \Delta P$ covers $B \cup \{v\}$ where $v \in A \setminus B$ as required.

2. Let $\mathcal{M}_1 = (V, \mathcal{I}_1)$ be the matching matroid, i.e. a set $U \subseteq V$ is in $\mathcal{I}_1$ if there exists a matching covering U. Moreover, let $\mathcal{M}_2 = (V, \mathcal{I}_2)$ be the partition matroid where $U \in \mathcal{I}_2$ if $|U \cap S| \le s$ and $|U \cap T| \le t$ and $U \subseteq S \cup T$. Observe that the basis of $\mathcal{M}_2$ has size $s + t$ and every independent set of $\mathcal{M}_2$ of this size must contain exactly s vertices from S and t vertices from T. Now, if there exists a common independent set of the two matroids of size $s + t$ then we answer yes otherwise no. Consider when we answer yes and let X be the common independent set of size $s + t$. Since $X \in \mathcal{I}_2$, we have $|X \cap S| = s$ and $|X \cap T| = t$ and since $X \in \mathcal{I}_1$ there is a matching covering X. Moreover, suppose there is such a matching. Pick any set $X_1 \subseteq S$ of size s that is covered by this matching and $X_2 \subseteq T$ of size t that is covered by this matching. Then $X_1 \cup X_2$ is a common independent set of the two matroids.