ACO Comprehensive Exam Fall 2022

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1 Design and Analysis of Algorithms

In a *combinatorial auction* there is a set N of $n = |N|$ bidders and a set M of $m = |M|$ items. Bidder $i \in N$ has a *monotone* valuation $v_i(\cdot)$ where $v_i(S)$ is their value for item set $S \subseteq M$ (here "monotone" means that $v_i(S \cup T) \geq v_i(S)$ for all $S, T \subseteq M$). The goal of this problem is to find a disjoint set of subsets where bidder i gets subset A_i of items ($A_i \cap A_k = \emptyset$ for $i \neq k$) to maximize the total *welfare* $\sum_{i \in N} v_i(A_i)$.

1. (Configuration LP, 2 points) Prove that the value of the following linear program (called the *configuration LP*) gives an upper bound on the total welfare of the optimal allocation.

$$\begin{aligned} \max \quad & \sum_{i \in N} \sum_{S \subseteq M} v_i(S) \cdot x_{i,S} \\ \text{s.t.} \quad & \forall i \in N, \quad \sum_{S \subseteq M} x_{i,S} = 1 \\ & \forall j \in M, \quad \sum_{S \ni j} \sum_{i \in N} x_{i,S} \leq 1 \\ & \forall i \in N, \forall S \subseteq M, \quad x_{i,S} \geq 0 \end{aligned}$$

2. (XOS Function, 3 points) A monotone set function $v(\cdot) : 2^M \to \mathbb{R}_{\geq 0}$ is called an *XOS function* if there exist monotone linear set functions $a_k(\cdot) : 2^M \to \mathbb{R}_{\geq 0}$ s.t. for all $S \subseteq M$ we have $v(S) = \max_k a_k(S)$ (i.e., $v(\cdot)$ can be written as the maximum of linear functions where a function $a_k(\cdot)$ is linear if it satisfies $a_k(S \cup T) = a_k(S) + a_k(T)$ for all disjoint $S, T \subseteq M$). Given a set $S \subseteq M$, prove that if we select a random subset $R \subseteq S$ from a probability distribution that contains each item in S with probability at least p (different items could be correlated), then the expected value of $v(R)$ is at least $p \cdot v(S)$.

3. (Rounding) Suppose we are given an optimal (fractional) solution $x^*_{i,S}$ to the configuration LP[1]. To"round" this fractional solution to integral allocations A_i, each bidder

[1] This can be computed in polynomial time using a "demand oracle" but we will assume that it is given.

$i \in N$ first chooses a random *tentative* item set T_i independent of other bidders, where $T_i = S$ with probability $x^*_{i,S}$ (the LP constraint $\sum_{S \subseteq M} x^*_{i,S} = 1$ ensures that this is a valid probability distribution). Since in this tentative allocation an item j might appear in multiple tentative sets, in the final allocation $\{A_i\}_i$ we allocate each item $j \in M$ to one of the tentative bidders (i.e., bidders i with $j \in T_i$) chosen uniformly at random.

(a) (3 points) Prove that conditioned on T_i, bidder i receives each item $j \in T_i$ with at least a constant probability, where the probability is taken over the random tentative sets T_k chosen by other bidders $k \neq i$.

(b) (2 points) Using (2), prove that if all valuations v_i are monotone XOS then the expected welfare of this rounded solution is at least a constant fraction of the optimal LP value $\sum_{i \in N} \sum_{S \subseteq M} v_i(S) \cdot x^*_{i,S}$, and so we get a constant factor approximation to the optimal welfare.

2 Combinatorial Optimization

Let $\mathcal{M} = (U, \mathcal{I})$ be a matroid with rank function $r : 2^U \to \mathbb{R}$ and let B and B' be two disjoint bases of $\mathcal{M}$. Let Y_1 and Y_2 be a partition of B. The problem is to prove the following statement:

There exists a partition Z_1 and Z_2 of B' such that $Y_1 \cup Z_1$ and $Y_2 \cup Z_2$ are both bases of $\mathcal{M}$.

To show this statement, prove the following steps (or give an alternative direct proof).

1. (1 point) We can assume without loss of generality that $U = B \cup B'$.
2. (2 points) Let $\mathcal{M}_1 = (\mathcal{M} \setminus Y_1)/Y_2$ and $\mathcal{M}_2 = (\mathcal{M}^\star \setminus Y_1)/Y_2$. Here $\mathcal{M}^\star$ is the dual matroid of $\mathcal{M}$ and $\mathcal{M}/Y$ denotes the matroid obtained by contracting elements in Y. What are the rank functions of $\mathcal{M}_1$ and $\mathcal{M}_2$ and, in particular, what are the ranks of these matroids?
3. (5 points) Show that there is a common independent set Z of size $|Y_1|$ of both these matroids.
4. (2 points) Show that $Z_2 = Z$ suffices to prove the statement.

3 Probabilistic Combinatorics

(10 points)
For a graph G, let $\mathsf{maxcut}(G)$ denote the maximum number of edges in a cut in G. Let $G \sim \mathbb{G}(n,p)$ be the Erdős–Rényi random graph with edge probability $p = p(n) \in [0,1]$ (so the edge probability is a function of n). Show that

$$\left| \mathbb{E}[\mathsf{maxcut}(G)] - \frac{pn^2}{4} \right| = O(\sqrt{p}\, n^{3/2}).$$

(The implied constants in the big-O notation should not depend on p.)
Remark. You may receive partial credit if you prove the result for only some range of values of p.

4 Solution: Algorithms

1. (a) Since this is a maximization LP, to prove the upper bound it suffices to show a feasible solution to the LP with value equal to the optimal welfare. Suppose in the optimal allocation bidder i gets item set A_i^* and the optimal welfare is $\sum_{i \in N} v_i(A_i^*)$. Now consider the fractional solution $x'_{i,S}$ where $x'_{i,S} = 1$ if $S = A_i^*$ and $x'_{i,S} = 0$ otherwise. This xi, S' is feasible for the configuration LP since by definition $\sum_{S \subseteq M} x'_{i,S} = 1$ for all $i \in N$ and since $\{A_i^*\}_i$ is an item partitioning we have $\sum_{S \ni j} \sum_{i \in N} x'_{i,S} \leq 1$ for all $j \in M$. The objective value of this solution is $\sum_{i \in N} \sum_{S \subseteq M} v_i(S) \cdot x'_{i,S} = \sum_{i \in N} v_i(A_i^*)$, which equals the optimal welfare.

   (b) We know $v(S) = \max_k a_k(S)$. Consider the linear function $a_\ell(\cdot)$ that achieves this maximum for S, i.e., $a_\ell(S) = v(S)$. We know by definition of XOS function that $v(T) \geq a_\ell(T)$ for every set $T \subseteq M$. Hence, to prove that $\mathbb{E}[v(R)] \geq p \cdot v(S)$, it suffices to show that $\mathbb{E}[a_\ell(R)] \geq p \cdot v(S) = p \cdot a_\ell(S)$. This last inequality is true by linearity of expectation since $a_\ell(\cdot)$ is a linear function and each element in S appears in R with probability at least p.

   (c) i. The expected number of bidders $k \neq i$ that contain item $j \in T_i$ in their tentative set equals $\sum_{k \neq i} \sum_{S \ni j} x^*_{kS} \leq 1$. So, by Markov's inequality, the probability that at least 2 bidders contain item j is at most $1/2$, so with probability at least $1/2$ at most one bidder $k \neq i$ contains item j, in which case i receives item j with probability at least $1/2$ in the uniformly random allocation. Overall, bidder i receives item $j \in T_i$ with probability at least $Pr[\leq 1$ tentative bidder $k \neq i$ for $j] \times Pr[i$ gets item $j \mid \leq 1$ tentative bidder $k \neq i$ for $j] \geq 1/2 \times 1/2 = 1/4$.

   ii. We first observe that if each bidder is assigned the random tentative set T_i, the expected welfare of bidder i equals $\mathbb{E}[v_i(T_i)] = \sum_{S \subseteq M} x^*_{i,S} v_i(S)$, and the total expected welfare equals the optimal LP value. However, $\{T_i\}_i$ is not a valid allocation since an item $j \in M$ might appear in multiple T_i. In our final allocation $\{A_i\}_i$ we uniformly randomly allocate any item $j \in T_i$ to one of the tentative bidders. So, by (2), to prove that we get least a constant fraction of the LP value, it suffices to prove after conditioning on T_i that bidder i receives each item $j \in T_i$ with at least $1/4$ probability, which we have proved above.

5 Solution: Combinatorial Optimization

1. Removing elements not in $B \cup B'$ does not affect the statement of the result.

2. First observe that the ground set of both $\mathcal{M}_1$ and $\mathcal{M}_2$ are exactly B'. For any set $Z \subseteq B'$, we have

$$r_1(Z) = r_{\mathcal{M}\setminus Y_1}(Z \cup Y_2) - r_{\mathcal{M}\setminus Y_1}(Y_2) = r(Z \cup Y_2) - r(Y_2) = r(Z \cup Y_2) - |Y_2|.$$

Similarly, we have

$$\begin{aligned} r_2(Z) &= r_{\mathcal{M}^\star\setminus Y_1}(Z \cup Y_2) - r_{\mathcal{M}^\star\setminus Y_1}(Y_2) \\ &= r_{\mathcal{M}^\star}(Z \cup Y_2) - r_{\mathcal{M}^\star}(Y_2) \\ &= |Z \cup Y_2| + r(U \setminus (Y_2 \cup Z)) - r(U) - (|Y_2| + r(U \setminus Y_2) - r(U)) \\ &= |Z| + r(Y_1 \cup (B' \setminus Z)) - r(Y_1 \cup B') \\ &= r(Y_1 \cup (B' \setminus Z)) - |B' \setminus Z| \end{aligned}$$

where we have used the formula for the rank function of a contracted matroid, dual matroid and Observe that $r_1(B') = r(B' \cup Y_2) - |Y_2| = |B'| - |Y_2| = |Y_1|$ where we use the fact that $|Y_1| + |Y_2| = |B| = |B'|$.

Also, we have $r_2(B') = r(Y_1) = |Y_1|$. Thus the rank of both matroids is $|Y_1|$.

3. We show there is a common independent set of size $|Y_1|$ for these matroids. The maximum size of the common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$ is exactly the $\min_{Z\subseteq B'} r_1(Z) + r_2(B' \setminus Z)$. But for any $Z \subseteq B'$ we have

$$\begin{aligned} r_1(Z) + r_2(B' \setminus Z) &= r(Z \cup Y_2) - |Y_2| + r(Y_1 \cup Z) - |Z| \\ &\geq r(Z \cup Y_1 \cup Y_2) + r(Z) - |Y_2| - |Z| \\ &= |Y_1| + |Y_2| + |Z| - |Y_2| - |Z| \\ &= |Y_1| \end{aligned}$$

as required.

4. Let Z be the maximum common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$. We claim $Y_2 \cup Z$ is a basis of $\mathcal{M}$. Indeed we have $Z \in \mathcal{M}_1$ implies that $Z \cup Y_2$ is independent in $\mathcal{M} \setminus Y_1$ and thus in $\mathcal{M}$. Moreover, Z is independent in $\mathcal{M}_2$ and thus $Z \cup Y_2$ is independent in $\mathcal{M}^\star \setminus Y_1$ and thus in $\mathcal{M}^\star$. In particular $U \setminus (Z \cup Y_2) = Y_1 \cup (U \setminus Z)$ is independent in $\mathcal{M}$ as required.

6 Solution: Probabilistic Combinatorics

To establish a lower bound on $\mathbb{E}[\mathsf{maxcut}(G)]$, we recall that $\mathsf{maxcut}(G) \geq |E(G)|/2$ for every graph G, and hence

$$\mathbb{E}[\mathsf{maxcut}(G)] \;\geq\; \frac{1}{2}\mathbb{E}[|E(G)|] \;=\; \frac{p}{2}\binom{n}{2} \;\geq\; \frac{pn^2}{4} - O(pn).$$

Since $pn \leq \sqrt{p}n^{3/2}$, this gives the right lower bound.
In the sequel we bound $\mathbb{E}[\mathsf{maxcut}(G)]$ from above.
Case 1: $p \leq 1600/n$. (Of course, 1600 here is just an arbitrary large constant.) In this case we make the trivial observation that $\mathsf{maxcut}(G) \leq |E(G)|$, which implies that

$$\mathbb{E}[\mathsf{maxcut}(G)] \;\leq\; \mathbb{E}[|E(G)|] \;=\; p\binom{n}{2} \;=\; O(pn^2).$$

It follows that the desired upper bound holds, since $pn^2 \leq 40\sqrt{p}n^{3/2}$ for $p \leq 1600/n$.
Case 2: $p > 1600/n$. For a partition $V(G) = A \sqcup B$, let $\mathsf{e}(A, B)$ denote the number of edges of G joining A to B. Then $\mathsf{maxcut}(G)$ is the maximum of $\mathsf{e}(A, B)$ taken over all partitions of $V(G)$.
Claim. $\mathbb{P}\left[\mathsf{maxcut}(G) > \frac{pn^2}{4} + 10\sqrt{p}\,n^{3/2}\right] \;<\; e^{-50n}$.
Proof. Consider any partition $V(G) = A \sqcup B$. Notice that $\mathsf{e}(A, B)$ is a binomial random variable with $|A||B|$ trials and success probability p. Since $|A||B| \leq n^2/4$, the Chernoff bound yields

$$\mathbb{P}\left[\mathsf{e}(A, B) > (1+\delta)\frac{pn^2}{4}\right] \;\leq\; \exp\left(-\frac{\delta^2}{3}\frac{pn^2}{4}\right) \quad \text{for all } 0 \leq \delta \leq 1.$$

Taking $\delta = 40/\sqrt{pn}$ (note that $\delta < 1$ since $p > 1600/n$) yields

$$\mathbb{P}\left[\mathsf{e}(A, B) > \frac{pn^2}{4} + 10\sqrt{p}\,n^{3/2}\right] \;\leq\; \exp\left(-\frac{400}{3}n\right) \;<\; e^{-100n}.$$

There are 2^{n-1} ways to partition $V(G)$ into two subsets, so the union bound gives

$$\mathbb{P}\left[\mathsf{maxcut}(G) > \frac{pn^2}{4} + 10\sqrt{p}\,n^{3/2}\right] \;\leq\; 2^{n-1} \cdot e^{-100n} \;<\; e^{-50n},$$

as desired. □
Since $\mathsf{maxcut}(G) \leq n^2/4$ for all G, we can use the above claim to write

$$\mathbb{E}[\mathsf{maxcut}(G)] \;\leq\; \frac{pn^2}{4} + 10\sqrt{p}\,n^{3/2} \;+\; \frac{n^2}{4} \cdot e^{-50n} \;=\; \frac{pn^2}{4} + O(\sqrt{p}\,n^{3/2}),$$

which completes the solution.