ACO Comprehensive Exam Fall 2024

Aug 16, 2024

1 Design and Analysis of Algorithms

Given an unweighted, undirected simple graph $G = (V, E)$ and two nodes $s \neq t \in V$, the min s-t cut problem is find a subset of edges $S \subseteq E$ of smallest size $|S|$ such that the graph $(V, E \setminus S)$ has no path from s to t. Consider the following LP to find the min s-t cut:

$$\min \sum_{e \in E} x_e \quad \text{subject to}$$
$$\forall \text{paths } P \text{ from } s \text{ to } t \text{ in } G, \quad \sum_{e \in P} x_e \geqslant 1$$
$$\forall e \in E, \quad x_e \geqslant 0$$

1. (3 points) Prove that the optimal objective value of this LP is at most the min s-t cut value. Give a polynomial-time separation oracle for this LP.
2. (3 points) Given an optimum solution x_e^* for $e \in E$ to the above LP, we will now round it to find an s-t cut of size at most $\sum_{e \in E} x_e^*$.

   Consider the graph $G = (V, E)$ with length of each edge e being x_e^*. These edge weights induce a shortest path metric on V. For $r \geqslant 0$, let $B_r \subseteq V$ denote all the vertices at distance at most r from node s. Show that for a uniform random $r \in [0, 1]$, the expected number of edges in the cut $(B_r, V \setminus B_r)$ is at most $\sum_{e \in E} x_e^*$.
3. (2 points) Show how we can derandomize the above procedure to obtain an s-t cut of size at most $\sum_{e \in E} x_e^*$.
4. (2 points) Write the dual LP of the above LP. Name the algorithmic problem that this dual LP corresponds to and interpret the dual variables.

2 Combinatorial Optimization

Let K be a complete graph with vertex set V and $T \subseteq V$ be a subset of vertices with even cardinality and let k be a positive integer. Let H be an edge-disjoint union of any k T-joins of K. We would like to show the following statement:

(*) For any $s \in T$, there exists a vertex $v \in V$ such that there are at least k-edge disjoint paths from s to v in H.

Prove the following statements below to infer the statement above.

1. (1 point) Given $s \in T$, it is sufficient to find a vertex $v \in V$ such that the minimum s-v cut in H is at least k.
2. (2 points) Show that if $S \subseteq V$ is such that $|S \cap T|$ is odd, then $\delta_H(S) \geqslant k$.
3. (1 point) Consider the Gomory-Hu tree of H and show that it is sufficient to find an edge of the Gomory-Hu tree incident at s that has weight at least k.
4. (6 points) Using the Gomory-Hu tree or otherwise, show that there exists a vertex v such that minimum s-v cut in H is at least k.

3 Probabilistic Combinatorics

Problem. Let $p = p(n) \in [0,1]$ and let $G \sim \mathbb{G}(n,p)$. Let D be the random variable equal to the largest size of a set of vertex-disjoint triangles in G. Prove the following statements:

(a)[4 points] If $p \ll n^{-2/3}$, then $D = o(n)$ with high probability.

(b)[6 points] If $p \gg n^{-2/3}$, then $D = (1 - o(1))(n/3)$ with high probability. (Hint: use Janson's inequality.)