ACO Comprehensive Exam Spring 2026

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1 Algorithms

In the *Maximum Directed Cut* problem, we are given a **directed** graph $G = (V, E)$ with no self-loops and edge weights $w_{ij} \geq 0$ for $(i, j) \in E$, and the goal is to partition the vertices into $(S, V \setminus S)$ so as to maximize the total weight of directed edges going from S to $V \setminus S$. Our goal is a 1/2-approximation algorithm for this problem.

1. (1 point) Suppose each vertex is independently chosen to be in S with probability 1/2 and in $V \setminus S$ otherwise. What is the probability that any specific directed edge (u, v) is in the directed cut?

2. (3 points) Write an Integer Program and the corresponding LP relaxation for the problem, where variable z_{uv} indicates whether a directed edge (u, v) is in the cut (i.e., $u \in S$ and $v \notin S$), and variable x_u indicates whether a vertex u is in S.

   Your LP formulation should include, along with other constraints, the following bounds:

   $$0 \leq x_u \leq 1 \quad \text{for all } u \in V, \qquad 0 \leq z_{uv} \leq 1 \quad \text{for all } (u, v) \in E.$$

   Show that any integer solution of your LP corresponds to a directed cut.

3. (2 points) Show that the integrality gap of this relaxation (i.e., the ratio of the LP optimum to the integer optimum) approaches 2 as the size of the graph goes to infinity.

4. (4 points) Let z^* and x^* denote the optimal fractional solution to the LP relaxation. Consider the randomized rounding scheme that sets $x_u = 1$ with probability $\frac{1}{4} + \frac{x_u^*}{2}$ and $x_u = 0$ otherwise (i.e., vertex u is selected independently into S with this probability). Show that this randomized algorithm achieves a 1/2-approximation in expectation.

   (Hint: for each edge (u, v), derive an expression for the probability that the rounding algorithm includes it in the cut, and then show that it is at least $z_{uv}^*/2$.)

2 Graph Theory

In an undirected graph $G = (V, E)$, each edge $\{u, v\} \in E$ corresponds to two oriented edges: uv and vu. For any positive integer k, we say that a undirected graph G admits a **k-flow** if there there exists a function f from the oriented edges of G to the set of integers, such that for any edge $\{u, v\}$ of G, $f(uv) = -f(vu) \neq 0$, $|f(uv)| < k$, and $\sum_{v \in N(u)} f(uv) = 0$ for $u \in V(G)$, where $N(u)$ is the set of vertices adjacent to u in G.

1. (7 points) Prove that a cubic graph admits 4-flow if and only if it is 3-edge-colorable.

2. (3 points) Provide an infinite family of cubic graphs that do not admit 4-flows.

3 Linear Inequalities

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a nonempty polyhedron defined by a matrix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Let

$$Q = \{y \in \mathbb{R}^n : \exists u \in \mathbb{R}^m, v \in \mathbb{R}^m \text{ s.t. } u \geq 0, v \leq 0, y = A^\top u = A^\top v, u^\top b = v^\top b\}.$$

1. (2 points) Show there exists a linear subspace L such that $P \subseteq L + x_0$ for some $x_0 \in P$ and $dim(P) = dim(L)$.
2. (3 points) Fix any $y \in Q$. Show that for any $x \in P$, $y^\top x$ is constant.
3. (5 points) Show that $dim(P) + dim(Q) = n$. *(Hint: use strong duality.)*

4 Solutions

4.1 Algorithms

1. (a) For any directed edge (u, v),

$$\Pr[(u,v) \text{ is cut}] = \Pr[u \in S] \Pr[v \notin S] = \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}.$$

   (b) Let $x_u = 1$ if $u \in S$ and 0 otherwise, and let $z_{uv} = 1$ if the directed edge (u, v) is cut and 0 otherwise.

   The integer program is:

$$\max \sum_{(u,v)\in E} w_{uv} z_{uv}$$

   subject to, for all $(u, v) \in E$,

$$z_{uv} \le x_u, \qquad z_{uv} \le 1 - x_v,$$
$$x_u \in \{0,1\}\ \forall u \in V, \qquad z_{uv} \in \{0,1\}\ \forall (u,v) \in E.$$

   The LP relaxation replaces the integrality constraints by:

$$0 \le x_u \le 1\ \forall u \in V, \qquad 0 \le z_{uv} \le 1\ \forall (u,v) \in E.$$

   In any integer solution, we take $S = \{u : x_u = 1\}$. We have $x_v = 0$ for all $v \in V \setminus S$. Then for any u, v we have $z_{uv} = 0$ if $x_u = 0$ or $x_v = 1$, thus all edges for which $z_{uv} = 1$ have $u \in S, v \in V \setminus S$.

   (c) Consider the complete bidirected graph on $2n$ vertices with weight 1 for every directed edge. Then the optimal solution is a bipartition with equal sides. The value of the cut is n^2. Now consider the fractional solution that assigns 1/2 to every directed edge variable and to every vertex variable. Then the value of the fractional solution is $(2n)(2n-1)/2 = n(2n-1)$ and the integrality gap is $n(2n-1)/n^2 = 2 - (1/n)$ which approaches 2 as $n \to \infty$.

   (d) For any edge (u, v),

$$\Pr[(u,v) \text{ is cut}] = \left(\tfrac{1}{4} + \tfrac{x_u^*}{2}\right)\left(1 - \left(\tfrac{1}{4} + \tfrac{x_v^*}{2}\right)\right) = \left(\tfrac{1}{4} + \tfrac{x_u^*}{2}\right)\left(\tfrac{1}{4} + \tfrac{1-x_v^*}{2}\right).$$

   From the LP constraints, $z_{uv}^* \le x_u^*$ and $z_{uv}^* \le 1 - x_v^*$, and since $w_{uv} \ge 0$, we get

$$\Pr[(u,v) \text{ is cut}] \ge \left(\tfrac{1}{4} + \tfrac{z_{uv}^*}{2}\right)\left(\tfrac{1}{4} + \tfrac{z_{uv}^*}{2}\right).$$

   This probability is at least $z_{uv}^*/2$ because

$$\left(\tfrac{1}{4} + \tfrac{z_{uv}^*}{2}\right)\left(\tfrac{1}{4} + \tfrac{z_{uv}^*}{2}\right) - \tfrac{z_{uv}^*}{2} = \tfrac{1}{16}\left((1 + 2z_{uv}^*)^2 - 8z_{uv}^*\right) = \tfrac{1}{16}\left((1 - 2z_{uv}^*)^2\right) \ge 0.$$

   By linearity of expectation,

$$\mathbb{E}[\mathrm{ALG}] = \sum_{(u,v)\in E} w_{uv} \Pr[(u,v) \text{ is cut}] \ge \frac{1}{2} \sum_{(u,v)\in E} w_{uv} z_{uv}^* = \frac{1}{2}\,\mathrm{LP}^* \ge \frac{1}{2}\,\mathrm{OPT}.$$

   Thus the algorithm is a 1/2-approximation in expectation.

4.2 Graph Theory

Part 1. Let G be a cubic graph. First, suppose G admits a 4-flow. Then by a theorem of Tutte, we know that G admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow, say f. So for every edge $\{u, v\}$ of G, $f(uv) \in \{(1,0), (0,1), (1,1)\}$. For each vertex $u \in V(G)$, since $\sum_{v \in N(u)} f(uv) = (0,0)$ and u has degree 3, we see that each of $\{(1,0), (0,1), (1,1)\}$ appears exactly once on the three edges incident with u. Hence, f gives a 3-edge-coloring of G.

Now suppose G is 3-edge-colorable. Let M_1, M_2, M_3 denote the three color classes of a 3-edge-coloring of G. Then, since G is cubic, M_1, M_2, M_3 are pairwise disjoint perfect matchings in G. Hence $G[M_1 \cup M_2]$ consists of disjoint cycles and, thus, admits the $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow f_1 such that $f_1(uv) = (1,0)$ for all edges $\{u, v\} \in M_1 \cup M_2$. Similarly, $G[M_2 \cup M_3]$ admits the $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow f_2 such that $f_2(uv) = (0,1)$ for all edges $\{u, v\} \in M_2 \cup M_3$. Now it is straightforward to verify that $f := f_1 + f_2$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on G. By Tutte's theorem again, G admits a 4-flow.

Part 2. In view of Part 1, it suffices to provide an infinite family of cubic graphs that are not 3-edge-colorable.

First we observe that if G is a cubic graph and G is not 3-edge colorable, then the following operation produces a cubic graph G' which is also not 3-edge-colorable. Let u be an arbitrary vertex of G and u_1, u_2, u_3 be the neighbors of u. Let G' be obtained from $G - u$ by adding a triangle $v_1 v_2 v_3 v_1$ and the edges $\{u_i, v_i\}$ for $i \in [3]$. Then it is easy to show that any 3-edge-coloring of G' would give a 3-edge-coloring of G. So G' is not 3-edge-colorable.
Now start with the Petersen graph, which can be easily shown not 3-edge-colorable. Repeatedly applying the above operation, we obtain an infinite family of cubic graphs that are not 3-edge-colorable (and hence do not admit 4-flows).

4.3 Linear Inequalities

1. (a) Pick any $x_0 \in P$. Let $L = \text{aff}(P) - x_0$ where aff(P) is the affine hull of P. By definition,
$$\dim(P) = \dim(\text{aff}(P)) = \dim(L).$$

   (b) Fix $y \in Q$. Then there exist $u \geq 0$ and $v \leq 0$ such that
$$y = A^\top u = A^\top v, \qquad u^\top b = v^\top b.$$
   Fix $x \in P$, so $Ax \leq b$. Since $u \geq 0$,
$$y^\top x = (A^\top u)^\top x = u^\top (Ax) \leq u^\top b.$$
   Since $v \leq 0$, multiplying $Ax \leq b$, we obtain
$$v^\top (Ax) \geq v^\top b.$$
   But $v^\top (Ax) = (A^\top v)^\top x = y^\top x$, so
$$v^\top b \leq y^\top x \leq u^\top b.$$
   Because $u^\top b = v^\top b$, it follows that $y^\top x$ is equal to this common value for all $x \in P$.

   (c) Let $L^\perp$ denote the perpendicular space to L. We show that $Q = L^\perp$. Since $dim(L) + dim(L^\perp) = n$, we have the desired equality.
   $Q \subseteq L^\perp$. If $y \in Q$, then by part (b) the function $x \mapsto y^\top x$ is constant on P, hence on aff$(P) = x_0 + L$. Therefore $y^\top d = 0$ for all $d \in L$, so $y \in L^\perp$.
   Conversely, let $y \in L^\perp$. Since $x \in \text{aff}(P)$ implies $x - x_0 \in L$, we have $y^\top x = y^\top (x - x_0) + y^\top x_0 = y^\top x_0$. Thus $y^\top x$ is constant on aff(P) and hence on P, say $y^\top x = \beta$. Consider the linear program
$$\max\{y^\top x : Ax \leq b\}.$$
   It is feasible and bounded, so by strong duality there exists $u \geq 0$ such that
$$A^\top u = y, \qquad u^\top b = \beta.$$
   Similarly, the problem
$$\min\{y^\top x : Ax \leq b\}$$
   is feasible and bounded with optimal value β. Its dual yields $v \leq 0$ such that
$$A^\top v = y, \qquad v^\top b = \beta.$$
   Hence $y \in Q$, and therefore $L^\perp \subseteq Q$.