ACO Comprehensive Exam Spring 2026

Jan 9, 2026

1 Algorithms

In the *Maximum Directed Cut* problem, we are given a **directed** graph $G = (V, E)$ with no self-loops and edge weights $w_{ij} \geq 0$ for $(i, j) \in E$, and the goal is to partition the vertices into $(S, V \setminus S)$ so as to maximize the total weight of directed edges going from S to $V \setminus S$. Our goal is a 1/2-approximation algorithm for this problem.

1. (1 point) Suppose each vertex is independently chosen to be in S with probability 1/2 and in $V \setminus S$ otherwise. What is the probability that any specific directed edge (u, v) is in the directed cut?

2. (3 points) Write an Integer Program and the corresponding LP relaxation for the problem, where variable z_{uv} indicates whether a directed edge (u, v) is in the cut (i.e., $u \in S$ and $v \notin S$), and variable x_u indicates whether a vertex u is in S.

   Your LP formulation should include, along with other constraints, the following bounds:

   $$0 \leq x_u \leq 1 \quad \text{for all } u \in V, \qquad 0 \leq z_{uv} \leq 1 \quad \text{for all } (u, v) \in E.$$

   Show that any integer solution of your LP corresponds to a directed cut.

3. (2 points) Show that the integrality gap of this relaxation (i.e., the ratio of the LP optimum to the integer optimum) approaches 2 as the size of the graph goes to infinity.

4. (4 points) Let z^* and x^* denote the optimal fractional solution to the LP relaxation. Consider the randomized rounding scheme that sets $x_u = 1$ with probability $\frac{1}{4} + \frac{x_u^*}{2}$ and $x_u = 0$ otherwise (i.e., vertex u is selected independently into S with this probability). Show that this randomized algorithm achieves a 1/2-approximation in expectation.

   (Hint: for each edge (u, v), derive an expression for the probability that the rounding algorithm includes it in the cut, and then show that it is at least $z_{uv}^*/2$.)

2 Graph Theory

In an undirected graph $G = (V, E)$, each edge $\{u, v\} \in E$ corresponds to two oriented edges: uv and vu. For any positive integer k, we say that a undirected graph G admits a k-**flow** if there there exists a function f from the oriented edges of G to the set of integers, such that for any edge $\{u, v\}$ of G, $f(uv) = -f(vu) \neq 0$, $|f(uv)| < k$, and $\sum_{v \in N(u)} f(uv) = 0$ for $u \in V(G)$, where $N(u)$ is the set of vertices adjacent to u in G.

1. (7 points) Prove that a cubic graph admits 4-flow if and only if it is 3-edge-colorable.

2. (3 points) Provide an infinite family of cubic graphs that do not admit 4-flows.

3 Linear Inequalities

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a nonempty polyhedron defined by a matrix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Let

$$Q = \{y \in \mathbb{R}^n : \exists u \in \mathbb{R}^m, v \in \mathbb{R}^m \text{ s.t. } u \geq 0, v \leq 0, y = A^\top u = A^\top v, u^\top b = v^\top b\}.$$

1. (2 points) Show there exists a linear subspace L such that $P \subseteq L + x_0$ for some $x_0 \in P$ and $dim(P) = dim(L)$.
2. (3 points) Fix any $y \in Q$. Show that for any $x \in P$, $y^\top x$ is constant.
3. (5 points) Show that $dim(P) + dim(Q) = n$. *(Hint: use strong duality.)*