ACO Comprehensive Exam Fall 2025

Aug 14, 2025

1 Design and Analysis of Algorithms

Let $G = (V, E)$ be a connected, undirected graph. The goal of this problem is to design an "optimal" random walk to sample uniformly from the vertices of G. The random walk is only allowed to use the edges of the graph and the probability of using an edge in either direction must be equal (the walk is "symmetric").

Let $X \in \mathbb{R}^{|V|\times|V|}$ denote the weights assigned to the edges. Then we have

$$X \geq 0, \quad X_{ij} = 0 \; \forall ij \notin E, \quad X = X^\top, \quad X\mathbf{1} = \mathbf{1}. \tag{1}$$

1. (1 pt) Show that for any matrix satisfying (1), for the corresponding random walk on G, the uniform distribution on the vertices is stationary.
2. (2 pts) Show that the largest eigenvalue of any such X is 1.
3. (3 pts) Write down a convex program whose feasible region is the set of matrices satisfying (1) and whose objective is to minimize the second largest eigenvalue of X (second largest in absolute value). Prove that your program is convex. [You are not required to bound the second largest eigenvalue.]
4. (2 pts) Write down an equivalent SDP.
5. (2 pts) What is the complexity of solving the above convex programs using an efficient cutting plane method? State both the number of separation oracle queries and the time to solve each query.

2 Combinatorial Optimization

Let $G = (S, T, E)$ be a bipartite graph with bipartition S and T. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid with rank function $r : 2^S \to \mathbb{Z}$.

1. (2 pts) Show that $\mathcal{M}' = (E, \mathcal{I}')$ is a matroid where $\mathcal{I}' = \{F \subseteq E : d_F(v) \leqslant 1 \ \forall v \in S$ and the subset of S covered by F is in $\mathcal{I}\ \}$.

2. Consider the following functions $b : 2^E \to \mathbb{Z}$ and $p : 2^E \to \mathbb{Z}$ defined as follows. For every $F \subseteq E$, we have $b(F) = r(\{u \in S : \exists \{u, v\} \in F\})$ as the rank of the subset of S covered by F. For every $F \subseteq E$, let $p(F) = |\{v \in T : \delta(v) \subseteq F\}|$ be the number of nodes v in T for which every edge of G incident at v is in F.

   (a) (2 pts) Show that b is a submodular function and p is a supermodular function.

   (b) (3 pts) Suppose that for every $X \subseteq T$, we have the following generalized Hall's condition
   $$r(\Gamma_S(X)) \geqslant |X|,$$
   where $\Gamma_S(X)$ denotes the set of neighbors of X in S. Show that $b(F) \geqslant p(F)$ for each $F \subseteq E$.

3. (3 pts) Under the assumption in the previous part, apply the integer-valued version of the Discrete Sandwich Theorem to p and b and show that it implies that G has a matching M covering T such that $M \in \mathcal{I}'$.

3 Probabilistic Combinatorics

Let $p(n), q(n), r(n)$ be three functions $\mathbb{N} \to [0, 1]$ satisfying $p(n) + q(n) + r(n) = 1$ and $p(n) \geq q(n) \geq r(n) \geq n^{-1.9}$. Let χ be a random 3-coloring of the edges of K_n where each edge is independently colored red with probability $p(n)$, blue with probability $q(n)$, and green with probability $r(n)$. A *rainbow triangle* in χ is a triangle with one edge of each color.

Find (with proof) a function $t(n)$ such that:

(a) (3 pts) If $p(n)q(n)r(n) = o(t(n))$, then w.h.p. χ has no rainbow triangles.

(b) (5 pts) If $p(n)q(n)r(n) = \omega(t(n))$, then w.h.p. χ has at least one rainbow triangle.

(c) (2 pts) Finally, for your choice of $t(n)$, exhibit functions $p(n), q(n), r(n)$ satisfying $p(n) + q(n) + r(n) = 1$ and $p(n)q(n)r(n) = \omega(t(n))$, but NOT $p(n) \geq q(n) \geq r(n) \geq n^{-1.9}$, for which w.h.p. χ has no rainbow triangles.

Solutions.

Design and Analysis of Algorithms

1. Since $X^\top \mathbf{1} = X\mathbf{1} = \mathbf{1}$, the all-ones vector is stationary.

2. By the Perron-Frobenius theorem, the largest eigenvalue has an all positive eigenvector v. Suppose $Av = \lambda v$ and $\lambda > 1$. Then, note that $\|Av\|_\infty \leq \|v\|_\infty$ while $\|\lambda v\|_\infty = \lambda\|v_\infty\| > \|v\|_\infty$, a contradiction.

3. We can simply write $\min \lambda(X), X$ satisfies (1), where $\lambda(X)$ denotes the second largest eigenvalue of X in magnitude. To see this is convex, it is the same as the larger of:

$$\max\{v^\top Xv, : \forall v^\top \mathbf{1} = 0, \|v\| = 1\}, \qquad -\min\{v^\top Xv, : \forall \|v\| = 1\}$$

   where the first expression is the second largest eigenvalue and the latter expression is the negative of the smallest eigenvalue. This shows that the objective is the maximum of a family of linear functions (one for each v from the first set and for each v from the second set) and hence is convex.

4. We can rewrite the program as

$$\begin{aligned} &\min y \\ &X \text{ satisfies } (1) \\ X - \frac{1}{n}\mathbf{1}\mathbf{1}^\top &\preccurlyeq yI \\ X - \frac{1}{n}\mathbf{1}\mathbf{1}^\top &\succeq -yI \end{aligned}$$

5. The cutting plane method (using ellipsoid or center-of-gravity) takes $O(|V|^2 \log(R/\varepsilon))$ separation queries to solve to within ε error. The feasible region is contained in a ball of radius $\sqrt{|V|}$ since the Frobenius norm of X is at most $\sqrt{|V|}$. Each separation call can be implemented in time $O(|V|^\omega)$, the time to find the second largest (in magnitude) eigenvalue/eigenvector of a symmetric, real matrix.

Combinatorial Optimization

(a) Clearly, the containment property is satisfied. Let $F, F' \in \mathcal{I}'$ such that $|F'| > |F|$. Let Y, Y' be the set of vertices in S covered by F and F' respectively. $|Y'| = |F'| > |F| = |Y|$. Since, $Y, Y' \in \mathcal{I}$, there exists a $y \in Y' \setminus Y$ such that $Y + y \in \mathcal{I}$. Let $(y, x) \in F'$. Clearly, $(y, x) \notin F$. Thus $F \cup \{y, x\}$ satisfies the conditions to be in $\mathcal{I}'$.

(b) (i) Let $F \subseteq E$ and $e, f \in E \setminus F$. Let $e = \{x, y\}$ with $y \in S$, $f = \{x', y'\}$. No with $y' \in S$ and Y be the endpoints of F in S. Now we have

$$b(F+e) - b(F) = r(Y+y) - r(Y) \geqslant r(Y+y+y') - r(Y+y') = b(F+e+f) - b(F+f)$$

where the inequality follows from submodularity of r. This shows submodularity of b.

Let $F \subseteq E$ and $e, f \in E \setminus F$. Consider $p(F+e) - p(F)$. If $p(F+e) - p(F) = 1$, it implies that all remaining edges incident at endpoint of e in T are already in F. In that case, we must have $p(F+f+e) - p(F+f) = 1$. Else, we must have $p(F+e) - p(F) = 0$ and $p(F+e+f) - p(F+f) \geqslant 0$. In either case, we have $p(F+e) - p(F) \leqslant p(F+e+f) - p(F+f)$ showing supermodularity of p.

(b) (ii) Let $F \subseteq E$. We show that $b(F) \geqslant p(F)$. Suppose not and there exists $F \subseteq E$ such that $b(F) < p(F)$. Take F to be minimal such F. Then if there exists an $e \in F$ incident at some vertex $t \in T$, then all edges incident at t must be in F. Else, $p(F \setminus e) = p(F) > b(F) \geqslant b(F \setminus e)$ and $F \setminus e$ is a smaller counterexample. Thus $F = \delta(X)$ for some $X \subseteq T$. Thus $p(F) = r(X)$ and $b(F) = b(\delta(X)) = |\Gamma_S(X)| \geqslant r(X)$ by the assumption.

(c) By the discrete sandwich theorem, there exists an integer-valued function $m : E \to \mathbb{Z}$ such that $b(F) \geqslant m(F) \geqslant p(F)$ for each subset $F \subseteq E$. Since $p \geqslant 0$, we must have $m \geqslant 0$. Since $b(\{e\}) \leqslant 1$, we have $m(e) \leqslant 1$ for each $e \in E$. Consider $M = \{e \in E : m(e) = 1\}$. Pick m such that M is minimal. Note that $p(\delta(t)) = 1$ for each $t \in T$. Hence, we have $\sum_{e \in \delta(t)} m(e) \geqslant 1$. Due to the minimality of M, we must have equality for each $t \in T$. Thus we have $|T| = |M|$. Moreover, $r(X) = b(F) \geqslant m(F) = |T|$ where X is the set of endpoint of M in S. Thus there are at least $|T|$ distinct endpoints of edges in S formed by these $|T|$ edges. Therefore, M is a matching and $|X| = |T|$. Since $r(X) \geqslant |T| = |X|$, so $r(X) = |X|$ and we must have $X \in \mathcal{I}$, and therefore $M \in \mathcal{I}'$ as required.

Probabilistic Combinatorics

The threshold function is $t(n) = n^{-3}$. Let X be the number of rainbow triangles in χ. Observe that

$$\mathbb{E}[X] = \binom{n}{3} \cdot 3! \cdot pqr \sim n^3 pqr.$$

If $pqr = o(n^{-3})$ then $\mathbb{E}[X] \to 0$ so by Markov's inequality, part (a) holds.

For part (b), since $\mathbb{E}[X] \to \infty$ if $p(n)q(n)r(n) = \omega(n^{-3})$, by Chebyshev's inequality it suffices to show that $\operatorname{Var}[X^2] = o(\mathbb{E}[X]^2)$. Write $X = \sum_S X_S$ where $S \in \binom{[n]}{3}$ ranges through all triples of three vertices in $[n]$, and expand $\operatorname{Var}[X^2]$ to obtain

$$\operatorname{Var}[X^2] = \sum_{S,T} \mathbb{E}[X_S X_T] - \mathbb{E}[X_S]\mathbb{E}[X_T] \le \mathbb{E}[X] + \sum_{|S| \cap |T| = 2} \mathbb{E}[X_S X_T],$$

where we used that whenever $|S \cap T| \le 1$ the events X_S and X_T are independent. The term $\mathbb{E}[X]$ is automatically $o(\mathbb{E}[X]^2)$, so it suffices to show the last summation is as well. It contains $O(n^4)$ terms, each of which is equal to $p^2q^2r + p^2qr^2 + pq^2r^2 = O(p^2q^2r) = p^2q^2r^2 \cdot O(r^{-1})$. Thus,

$$\sum_{|S| \cap |T| = 2} \mathbb{E}[X_S X_T] \le O(n^4) \cdot (pqr)^2 \cdot O(r^{-1}) = O(n^{5.9}(pqr)^2) = o(\mathbb{E}[X]^2),$$

as desired.

For the last part, observe that if $r = n^{-2.5}$ and $p = q = \frac{1-r}{2}$, then $pqr = \omega(n^{-3})$ but w.h.p. χ has no green edges at all, and therefore no rainbow triangles.