LATTICE POINTS, ORIENTED MATROIDS, AND ZONOTOPES

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By

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SUMMARY

This thesis consists of three chapters on various topics in discrete geometry. The main theme of the first chapter is about lattice points, while the second and third chapters are on the subject of oriented matroids.

In the first chapter, we analyse the distribution of lattice points in a half-open parallelepiped. In particular, we give an explicit presentation of the linear span of the lattice points inside of a half-open integral parallelepiped, in terms of the edges which generate the parallelpiped.

In the second chapter, we study polyhedral realizations of oriented matroids. In particular, we study a polyhedral fan which plays the role of the Bergman fan for oriented matroids. We show that this fan is a subfan of the normal fan of a certain naturally defined polytope which we call the signed matroid polytope. We study the cones of this fan, and describe their extreme rays explicitly. In the case when the oriented matroid is uniform, we show that the face lattice of this fan is anti-isomorphic to the face poset of the cellular decomposition of a pseudosphere arrangement representing the oriented matroid.

In the third chapter, we revisit the problem of tilings of zonotopes by zonotopes. We give a new proof of one direction of the Bohne-Dress theorem, which states that zonotopal tilings of a zonotope arise from single element liftings of the oriented matroid associated to the zonotope. This proof is topological in nature, and the chirotope plays a central role. We also speculate on generalizations of the Bohne-Dress theorem to the nonrealizable setting.

Part I

Lattice points

CHAPTER 1

LATTICE POINTS IN A PARALLELEPIPED

Introduction

Let n be a positive integer and let Λ denote a lattice in $\mathbf{R}^n$ that contains the integer lattice $\mathbf{Z}^n$. We are interested in understanding the combinatorics of the lattice points of Λ inside the half-open cube

$$[0,1)^n := \{(x_1,\ldots,x_n) \in \mathbf{R}^n :\ 0 \leq x_i < 1 \text{ for all } i = 1,2,\ldots,n\}\,.$$

In general, questions about these points are difficult. For instance, if $\Lambda = \frac{1}{2}\mathbf{Z}^n$ and $u = (u_1,\ldots,u_n) \in \mathbf{R}^n$ has integral coordinates, then the problem of deciding if there exists a nonzero point in $\Lambda \cap [0,1)^n$ on the hyperplane $\{x \in \mathbf{R}^n : \langle u, x\rangle = 0\}$ is NP-complete. Indeed, it is straightforward to reduce SUBSET-SUM to this problem; such a point exists if and only if some integers in the multiset $\{u_1,\ldots,u_n\}$ sum to zero. As pointed out by Sebő in [1, p. 401], the well-known *Lonely Runner Conjecture [2]* can be stated as a problem about the existence of a lattice point in $\Lambda \cap [0,1)^n$ satisfying certain linear inequalities where the lattice Λ is generated by $\mathbf{Z}^n$ plus a rational vector $v \in \mathbf{R}^n$ encoding the speeds of the runners.

Our approach to understanding the lattice points in $\Lambda \cap [0,1)^n$ begins with a result that is commonly attributed to G. K. White [3] but was discovered independently by several others [4, 5]. It says that a tetrahedron T in $\mathbf{R}^3$ which has integral vertices but no other integral points must be "sandwiched" between two parallel lattice hyperplanes. More precisely, there exists an integral normal vector $u = (u_1, u_2, u_3)$ and an integer δ such that two of the vertices of T lie on the plane $\langle u, x\rangle = \delta$ and the other two lie on the plane $\langle u, x\rangle = \delta + 1$. We may assume that one of the vertices of the tetrahedron lies at the origin, so that there are three integral vectors v, v', v'' corresponding to the three edges of the tetrahedron incident to the origin. The tricky part of White's theorem is to show that, after applying a unimodular transformation (i.e. a linear transformation of $\mathbf{R}^n$ which fixes $\mathbf{Z}^n$), we may further assume that $v = (1,0,0)$, $v' = (0,1,0)$, and $v'' = (1,a,r)$ where a and r are positive integers such that $a < r$ and a is coprime to r. From there, the normal vector $u = (1,0,0)$ establishes the conclusion of the theorem.

The triples $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbf{R}^3$ such that $\lambda_1 v + \lambda_2 v' + \lambda_3 v'' \in \mathbf{Z}^3$ form a lattice $\Lambda \subset \mathbf{R}^3$ which contains the integer lattice $\mathbf{Z}^3$. Moreover, T contains a non-vertex integral point if and only if there exists some nonzero $\lambda \in \Lambda \cap [0,1)^3$ such that $\lambda_1 + \lambda_2 + \lambda_3 \leq 1$. Indeed, such a λ corresponds to a proper convex combination of at least two vertices of T. A short exercise shows that there are exactly r lattice points in $\Lambda \cap [0,1)^3$ and they are of the form $(\{k/r\}, \{ka/r\}, \{-k/r\})$ for $k = 0, 1, 2, \ldots, d-1$. Here $\{x\}$ denotes the fractional part of the real number x, the unique real number in $[0,1)$ congruent to x mod 1. In particular, we can view the emptiness of T as a consequence of the fact that the first two components of every nonzero $\lambda \in \Lambda \cap [0,1)^3$ sum to 1 and therefore the sum $\lambda_1 + \lambda_2 + \lambda_3$ exceeds 1.

More generally, if $\Lambda \subset \mathbf{R}^n$ is a lattice that contains $\mathbf{Z}^n$, then we can think of the presence of such complementary pairs of coordinates as a restriction on the extent to which the nonzero points in $\Lambda \cap [0,1)^n$ can deviate from the hyperplane $x_1 + \cdots + x_n = n/2$. Sebő asks in [1] about the most restrictive case, where *all* the nonzero lattice points in $\Lambda \cap [0,1)^n$ lie on this hyperplane. He conjectures that this can only happen if the coordinates can be grouped into $n/2$ pairs of complementary coordinates as above. More precisely, suppose Λ is a lattice in $\mathbf{R}^n$ generated by $\mathbf{Z}^n$ and the point $\frac{1}{r}(a_1, \ldots, a_n)$, where the a_i's are positive integers coprime to a positive integer r. Note that for every $(\lambda_1, \ldots, \lambda_n) \in \Lambda \cap [0,1)^n$, there exists an integer $0 \leq k < r$ such that $\lambda_i = \{ka_i/r\}$ for each $i = 1, 2, \ldots, n$. Sebő asks if the following statement is true:

Conjecture 1.1.1. *The equality*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = n/2$$

holds for all nonzero $\lambda \in \Lambda \cap [0,1)^n$ if and only if n is even and (after possibly reordering) $a_i + a_{i+1} = r$ for $i = 1, 3, 5, \ldots, n-1$.

In [1], Sebő proves the case $n = 4$ of his conjecture and uses it to deduce White's theorem.

It turns out, however, that Sebő's conjecture had already been established some years earlier by Morrison and Stevens in their paper [6] (see also [7]). Although they were also primarily interested in the case $n = 4$, their proof stands out as it easily extends to all positive even integers n. In [6], Morrison and Stevens use this result to derive a complete classification of three dimensional isolated terminal cyclic quotient singularities and four dimensional isolated Gorenstein terminal cyclic quotient singularities. The survey paper of Borisov [8] provides a nice description and some interesting number-theoretic applications of this problem.

In [9, Theorem 5.4], Reid proves a stronger version of Conjecture 1.1.1 that does not require the a_i's to be coprime to r. Given a lattice $\Lambda \subset \mathbf{R}^n$ generated by $\mathbf{Z}^n$ and a point $\frac{1}{r}(a_1, \ldots, a_n)$ where the a_i's are positive integers less then r, he finds a characterization for when all lattice points in $\Lambda \cap [0,1)^n$ lie in a given hyperplane through the origin. We show in Section 1.3 how to deduce Sebő's conjecture from Reid's result, known as the *Terminal Lemma*. In [9, Section 6], Reid shows how the Terminal Lemma can be systematically applied to obtain Mori's classification results on three dimensional terminal singularities found in [10].

Other variations of Conjecture 1.1.1 have found application in Ehrhart theory; in particular the problem of classifying lattice polytopes with a given h^*-polynomial. In [11], Batyrev and Hofscheier give a classification of all lattice polytopes whose

h^*-polynomial is of the form $h^*(t) = 1 + ct^k$ for some positive integers c, k in terms of particular linear codes. This work was further developed by Higashitani, Nill, and Tsuchiya in [12] in order to obtain a combinatorial description of Gorenstein polytopes with a trinomial h^*-polynomial. A key ingredient of these results is a version of Conjecture 1.1.1 applicable to lattices $\Lambda \subset \mathbf{R}^n$ containing $\mathbf{Z}^n$ with the property that the quotient group $\Lambda/\mathbf{Z}^n$ is isomorphic to the additive group of a finite field.

In this paper, we establish of a variation of Conjecture 1.1.1 which imposes no restrictions on the lattice $\Lambda \subset \mathbf{R}^n$ except that it must contain the integer lattice $\mathbf{Z}^n$. The paper is organized into six sections. Following the introduction, Section 1.2 outlines the basic notation and concepts we use. In Section 1.3, we state our main theorem, which directly generalizes Reid's Terminal Lemma by dropping the assumption $\Lambda/\mathbf{Z}^n$ must be cyclic. From our theorem we deduce a formula for the dimension of the linear span of the points in $\Lambda \cap [0,1)^n$. We also state a natural generalization of Conjecture 1.1.1 when there are no assumptions on the group structure of $\Lambda/\mathbf{Z}^n$. Finally, we state the two main technical tools needed to prove our main theorem. Section 1.4 outlines the proof of our main theorem using these two tools, both of which are statements about an arbitrary additive finite abelian group G. Section 1.5 contains the proof of the first technical tool, which asserts that a specific collection of indicator functions defined on G is linearly independent. In Section 1.6 is the proof of the second technical tool, which gives a specific spanning set for the space of functions $f : G \to \mathbf{C}$ satisfying $f(-a) = -f(a)$ for all $a \in G$. At a high level, we mostly follow the path laid out by Reid in [9]. We differ somewhat in the details, however, by making liberal use of the results in [13, Section 9.2].

Background and notation

Notation

For $u = (u_1, \ldots, u_n) \in \mathbf{R}^n$ and $v = (v_1, \ldots, v_n) \in \mathbf{R}^n$, we let $\langle u, v\rangle = u_1v_1 + \cdots + u_nv_n$ denote the usual dot product. If v is a vector in a vector space with specified coordinates, then $\operatorname{supp}(v)$ denotes the set of coordinates i for which $v_i \neq 0$. For $x \in \mathbf{R}$, we define $\{x\}$ to be the unique real number in the half-open interval $[0, 1)$ in which $x - \{x\}$ is an integer. We frequently make use of the fact that for any $x \in \mathbf{R}$, $\{x\}+\{1-x\}$ equals 1 if $x \notin \mathbf{Z}$ and 0 otherwise. We define the first periodic Bernoulli function $B_1 : \mathbf{R} \to \mathbf{R}$ by

$$B_1(x) := \begin{cases} \{x\} - 1/2, & x \notin \mathbf{Z} \\ 0, & x \in \mathbf{Z}. \end{cases}$$

For $x + \mathbf{Z} \in \mathbf{R}/\mathbf{Z}$, we also define $B_1(x + \mathbf{Z}) := B_1(x)$.

For a finite group G, we denote the space of complex functions $f : G \to \mathbf{C}$ by $L^2(G)$ which forms a vector space under pointwise addition and comes with the inner product

$$\langle f, h\rangle = \frac{1}{|G|} \sum_{g\in G} f(g)\overline{h(g)}.$$

Character theory of finite abelian groups

We refer the reader to [14] and [15] for an introduction to the character theory of finite abelian groups, and record some key facts here. For a finite abelian group G, let $\widehat{G}$ denote the multiplicative group of homomorphisms $G \to \mathbf{C}^\times$ from G to the nonzero complex numbers. The group operation of $\widehat{G}$ is given by pointwise multiplication: $(\chi\psi)(g) := \chi(g)\psi(g)$ for each $g \in G$ and for each $\chi, \psi \in \widehat{G}$. The inverse of $\chi \in \widehat{G}$ satisfies $\chi^{-1}(g) = \overline{\chi(g)}$ for all $g \in G$; we therefore denote χ^{-1} by $\overline{\chi}$. Elements in $\widehat{G}$

are called *characters* of G, and they form an orthonormal basis of $L^2(G)$. There is an isomorphism $G \simeq \widehat{G}$ and we identify G with $\widehat{\widehat{G}}$ via the natural isomorphism which takes $g \in G$ to the point evaluation map $(\chi \mapsto \chi(g)) \in \widehat{\widehat{G}}$. For a subgroup K of G, let

$$K^{\perp} := \{\chi \in \widehat{G} : \chi(k) = 1 \text{ for all } k \in K\}$$

which is a subgroup of $\widehat{G}$. With the above identification of G and $\widehat{\widehat{G}}$, we have $K^{\perp\perp} := (K^{\perp})^{\perp} = K$.

We define $e : \mathbf{R}/\mathbf{Z} \to \mathbf{C}^{\times}$ to be the injective group homomorphism $x + \mathbf{Z} \mapsto \exp(2\pi i x)$. Since the additive group $\mathbf{R}/\mathbf{Z}$ embeds into the multiplicative group $\mathbf{C}^{\times}$ of nonzero complex numbers via the map $x + \mathbf{Z} \mapsto e(x)$, it follows that the additive group $H := \operatorname{Hom}_{\mathbf{Z}}(G, \mathbf{R}/\mathbf{Z})$ is isomorphic to the multiplicative group $\widehat{G}$ via the map $\phi \mapsto e \circ \phi$. In this paper it will typically be more convenient to state results and proofs in terms of H rather than $\widehat{G}$. However, we will sometimes take advantage of both the multiplicative and additive structure offered by $\mathbf{C}$ and work with $\widehat{G}$ instead.

Overview of results

We begin with our generalization of Reid's Terminal Lemma as claimed in the abstract. Let $\Lambda \subset \mathbf{R}^n$ be a lattice containing $\mathbf{Z}^n$. For $i = 1, 2, \ldots, n$, let $\pi_i : \Lambda/\mathbf{Z}^n \to \mathbf{R}/\mathbf{Z}$ denote the coordinate projection map sending $(\lambda_1, \ldots, \lambda_n) + \mathbf{Z}^n$ to $\lambda_i + \mathbf{Z}$. Observe that these maps are homomorphisms in the additive group $\operatorname{Hom}_{\mathbf{Z}}(\Lambda/\mathbf{Z}^n, \mathbf{R}/\mathbf{Z})$ under pointwise addition; thus, it makes sense to talk about $-\pi_i$ for each i. By restricting to the appropriate subspace of $\mathbf{R}^n$, we assume without loss of generality that $\ker \pi_i \neq \Lambda/\mathbf{Z}^n$ for any i.

Theorem 1.3.1 (Terminal Lemma, cf. [9, Theorem 5.4])**.** *Let* $u = (u_1, \ldots, u_n) \in \mathbf{R}^n$.

Then $\langle u, \lambda\rangle = 0$ *for every* $\lambda \in \Lambda \cap [0,1)^n$ *if and only if*

$$\sum_{\substack{i=1 \\ \pi_i = \pi_j}}^{n} u_i = \sum_{\substack{i=1 \\ \pi_i = -\pi_j}}^{n} u_i \tag{1.1}$$

and

$$\sum_{\substack{i=1 \\ \ker \pi_i = \ker \pi_j}}^{n} u_i = 0 \tag{1.2}$$

for each $j = 1, 2, \ldots, n$.

From this theorem several corollaries can be deduced. The first shows that the dimension of the span of the lattice points of Λ in the half-open unit cube $[0,1)^n$ can be computed explicitly in terms of the coordinate projection functions $\pi_i : \Lambda/\mathbf{Z}^n \to \mathbf{R}/\mathbf{Z}$.

Let $\mathcal{I}$ denote the equivalence classes of the equivalence relation on the coordinates $\{1, 2, \ldots, n\}$ in which $i \sim j$ in $\mathcal{I}$ if and only if $\pi_i = \pi_j$ or $\pi_i = -\pi_j$. Let $\mathcal{K}$ denote the equivalence classes of the equivalence relation on the coordinates $\{1, 2, \ldots, n\}$ where $i \sim j$ in $\mathcal{K}$ if and only if $\ker(\pi_i) = \ker(\pi_j)$. Note that $\mathcal{K}$ coarsens $\mathcal{I}$ since $\ker(\pi_i) = \ker(-\pi_i)$ for all i.

Corollary 1.3.2. *The dimension of* $\operatorname{span}(\Lambda \cap [0,1)^n)$ *is equal to* $\iota + \kappa$, *where*

$$\iota := |\{I \in \mathcal{I} : \pi_i \neq -\pi_i \textit{ for some (and hence all) } i \in I\}|$$

and

$$\kappa := |\{I \in \mathcal{K} :\ \pi_i = -\pi_j \textit{ for some (possibly equal) } i, j \in I\}|\,.$$

Proof. The distinct relations of the form (1.1) are in 1-1 correspondence with the equivalence classes $[i] \in \mathscr{I}$ such that $\pi_i \neq -\pi_i$. Note that in case $\pi_i = -\pi_i$, the relation (1.1) is trivial. Similarly, the distinct relations of the form (1.2) are in 1-1 correspondence with the equivalence classes of $\mathscr{K}$. The collection of all these relations are linearly independent except in the case when some $J \in \mathscr{K}$ does not contain any

Table 1.1: An illustration of Theorem 1.3.1 and Corollary 1.3.2 for the lattice Λ generated by $\mathbf{Z}^8$ and the two points $\lambda = \frac{1}{10}(1,9,3,7,1,1,3,5)$ and $\lambda' = \frac{1}{10}(2,8,6,4,1,1,3,0)$. The coordinate projection maps $\pi_i : \Lambda/\mathbf{Z}^8 \to \mathbf{R}/\mathbf{Z}$ are uniquely determined by the two numbers λ_i and λ'_i. In this example, $\iota = 4$ (corresponding to the classes I_1, I_2, I_3, I_4 in $\mathscr{I}$) and $\kappa = 2$ (corresponding to the classes J_1, J_3 in $\mathscr{K}$). By Corollary 1.3.2, the dimension of the linear span of $\Lambda \cap [0,1)^8$ is 6.

$\mathscr{K}$	J_1					J_2				J_3	
$\mathscr{I}$	I_1		I_2			I_3		I_4		I_5	
i	1	2	3	4		5	6	7		8	
λ_i	0.1	0.9	0.3	0.7		0.1	0.1	0.3		0.5	
λ'_i	0.2	0.8	0.6	0.4		0.1	0.1	0.3		0.0	
(1.1) relations	u_1	$-u_2$			$= 0$	u_5	$+u_6$		$= 0$	$u_8 - u_8$	$= 0$
			u_3	$-u_4$	$= 0$			u_7	$= 0$		
(1.2) relations	u_1	$+u_2$	$+u_3$	$+u_4$	$= 0$	u_5	$+u_6$	$+u_7$	$= 0$	u_8	$= 0$

coordinates i, j for which $\pi_i = -\pi_j$. In this situation, the relation (1.2) corresponding to J is already implied by the relations (1.1) corresponding to the equivalence classes $I \in \mathscr{I}$ contained in J. Thus, after excluding the relations corresponding to such $J \in \mathscr{K}$, we conclude that the space of $u \in \mathbf{R}^n$ in which $\langle u, \lambda \rangle = 0$ for all $\lambda \in \Lambda \cap [0,1)^n$ has dimension $n - \iota - \kappa$ and hence the dimension of the span of $\Lambda \cap [0,1)^n$ equals $\iota + \kappa$. $\square$

Observe that it is always true that

$$\lambda_1 + \cdots + \lambda_n + \mu_1 + \cdots + \mu_n = |\mathrm{supp}\,(\lambda)| = |\mathrm{supp}\,(\mu)|$$

for every pair $\lambda, \mu \in \Lambda \cap [0,1)^n$ for which $\lambda + \mathbf{Z}^n = -\mu + \mathbf{Z}^n$. This follows from the fact that for every $i = 1, 2, \ldots, n$, we have either $\lambda_i = 1 - \mu_i$ if both λ_i and μ_i are nonzero, or $\lambda_i = \mu_i = 0$ otherwise. The next corollary characterizes the situation where the "mass" of $\lambda + \mu$ is distributed as equally as possible between λ and μ for all such pairs λ, μ. It is a direct generalization of Sebő's Conjecture 1.1.1:

Corollary 1.3.3 (cf. [1, Conjecture 4.1], [7, Proposition 1.8])**.** *The equality*

$$\lambda_1 + \cdots + \lambda_n = \frac{|\mathrm{supp}\,(\lambda)|}{2} \tag{1.3}$$

holds for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda \cap [0,1)^n$ *if and only if there exists an involution* σ *of* $\{1, 2, \ldots, n\}$ *(i.e. a bijection satisfying* $\sigma = \sigma^{-1}$*) such that* $\lambda_i + \lambda_{\sigma(i)}$ *is an integer for all* $i = 1, 2, \ldots, n$ *and* $\lambda \in \Lambda$*.*

Example 1.3.4. If the coordinates of the points in Λ are all half-integral (i.e. $\Lambda \subset \frac{1}{2}\mathbf{Z}^n$), then Corollary 1.3.3 is trivial. Indeed, both the hypothesis and the conclusion always hold; for the conclusion we may take σ to be the identity map.

Example 1.3.5. If Λ is generated by $\mathbf{Z}^n$ and the point $\frac{1}{r}(a_1, a_2, \ldots, a_n)$, where the a_i's are positive integers coprime to r, then we recover Conjecture 1.1.1. Indeed, in this case $|\mathrm{supp}(\lambda)| = n$ for every nonzero $\lambda \in [0,1)^n \cap \Lambda$, and $\lambda_i + \lambda_{\sigma(i)} \in \mathbf{Z}$ for all $i = 1, 2, \ldots, n$ and $\lambda \in \Lambda$ if and only if $a_i + a_{\sigma(i)} = r$ for all $i = 1, 2, \ldots, n$. This follows from the fact that for every nonzero $\lambda \in \Lambda \cap [0,1)^n$ there exists $1 \leq k \leq r-1$ such that $\lambda_i = \{k a_i / r\} \neq 0$ for each $i = 1, 2, \ldots, n$.

Example 1.3.6. If $\Delta \subseteq \mathbf{R}^{n-1}$ is a lattice polytope, then the Ehrhart series of Δ is given by

$$\mathrm{Ehr}_\Delta(t) = \sum_{m \geq 0} \left| m\Delta \cap \mathbf{Z}^{n-1} \right| t^m = \frac{1 + h_1^* t + \cdots + h_{n-1}^* t^{n-1}}{(1-t)^n}$$

and the numerator of the right-hand side is called the h^**-polynomial* of Δ. If furthermore $\Delta = \mathrm{conv}(v_1, \ldots, v_n)$ is a simplex, where each $v_i \in \mathbf{Z}^{n-1}$, then it is known that h_k^* equals the number of $\lambda \in \Lambda_\Delta \cap [0,1)^n$ such that $\lambda_1 + \cdots + \lambda_n = k$ [16, Corollary 3.11]. Here $\Lambda_\Delta \subset \mathbf{R}^n$ denotes the dual lattice of the lattice generated by $(v_i, 1) \in \mathbf{Z}^n$ for $i = 1, 2, \ldots, n$; equivalently, the lattice of points $\lambda \in \mathbf{R}^n$ such that $\langle \lambda, (v_i, 1) \rangle \in \mathbf{Z}$ for all i.

Polytopes Δ with h^*-polynomial of the form $1 + h_k^* t^k$ for some positive k have been completely classified by Batyrev and Hofscheier [11]. They show that such a polytope Δ must be a simplex; therefore, the corresponding lattice Λ_Δ has the property that $\lambda_1 + \cdots + \lambda_n = k$ for all nonzero $\lambda \in \Lambda_\Delta \cap [0,1)^n$. It follows that the hypothesis of Corollary 1.3.3 applies to Λ_Δ, and the resulting involution σ appears in their classification. They also describe some properties of $\Lambda_\Delta/\mathbf{Z}^n$; for instance, $\Lambda_\Delta/\mathbf{Z}^n$ is isomorphic to the additive group of $\mathbf{F}_p^r$ for some prime p and integer r, and the integer k satisfies $(p^r - p^{r-1})n = 2k(p^r - 1)$.

Proof of Corollary 1.3.3. The "if" direction is an immediate consequence of the fact that, for every $x \in \mathbf{R}$, $\{x\} + \{-x\}$ equals 1 if $x \notin \mathbf{Z}$ and 0 otherwise.

For the "only if" direction, consider the lattice $\Lambda' \subset \mathbf{R}^{2n}$ which is generated by $\mathbf{Z}^{2n}$ and the image of the map $\Lambda \to \mathbf{R}^{2n}$ defined by

$$(\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n).$$

Let $\lambda \in \Lambda \cap [0,1)^n$ and let $\lambda' \in \Lambda' \cap [0,1)^{2n}$ be the unique integral translate in $[0,1)^{2n}$ of the image of λ under this map. Let $\mu \in \Lambda \cap [0,1)^n$ the the unique lattice point in $[0,1)^n$ which satisfies $\lambda + \mathbf{Z}^n = -\mu + \mathbf{Z}^n$. Then

$$\lambda'_1 + \cdots + \lambda'_n = \lambda'_{n+1} + \cdots + \lambda'_{2n}$$

since by assumption we have

$$\lambda'_1 + \cdots + \lambda'_n = \lambda_1 + \cdots + \lambda_n = \frac{|\mathrm{supp}(\lambda)|}{2}$$

and

$$\lambda'_{n+1} + \cdots + \lambda'_{2n} = \mu_1 + \cdots + \mu_n = \frac{|\mathrm{supp}(\mu)|}{2}$$

and we know by the preceding discussion that $|\mathrm{supp}(\mu)| = |\mathrm{supp}(\lambda)|$. If we let

$$u' = (\underbrace{1, \dots, 1}_{n}, \underbrace{-1, \dots, -1}_{n}) \in \mathbf{R}^{2n},$$

we get $\langle u', \lambda' \rangle = 0$ for each $\lambda' \in \Lambda' \cap [0, 1)^{2n}$. We may therefore apply Theorem 1.3.1 to obtain the equality

$$\left(\sum_{\substack{i=1 \\ \pi_i = \pi_j}}^{n} 1 \right) - \left(\sum_{\substack{i=1 \\ \pi_i = -\pi_j}}^{n} 1 \right) = \left(\sum_{\substack{i=1 \\ \pi_i = -\pi_j}}^{n} 1 \right) - \left(\sum_{\substack{i=1 \\ \pi_i = \pi_j}}^{n} 1 \right)$$

for each $j \in \{1, 2, \dots, n\}$, which simplifies to

$$\left| \{ i : \pi_i = \pi_j \} \right| = \left| \{ i : \pi_i = -\pi_j \} \right|.$$

We now construct our involution $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$. For each i such that $\pi_i = -\pi_i$, we set $\sigma(i) = i$. For each coordinate projection map π such that $\pi \neq -\pi$, we pair up each coordinate i such that $\pi_i = \pi$ with a unique coordinate j such that $\pi_j = -\pi$. Then, for each such pair (i, j), we set $\sigma(i) = j$ and $\sigma(j) = i$. Now let $\lambda \in \Lambda$ and let $i \in \{1, 2, \dots, n\}$. Then

$$\lambda_i + \lambda_{\sigma(i)} + \mathbf{Z} = \pi_i(\lambda) + (-\pi_i)(\lambda) = 0 + \mathbf{Z},$$

and hence $\lambda_i + \lambda_{\sigma(i)}$ is an integer. $\square$

The proof of Theorem 1.3.1, specifically the "only-if" direction, depends on the following two claims. The first is used to establish the relations (1.2) assuming the hypotheses of Theorem 1.3.1. The proof given in the next section relies on the Poisson summation formula for finite abelian groups.

Lemma 1.3.7. *Let G be a finite abelian group. For a subgroup K of G, let $\mathbf{1}_K \in$*

$L^2(G)$ *denote the indicator function of* K*. Then*

$$\{\mathbf{1}_K \ : \ K^\perp \text{ is a cyclic subgroup of } \widehat{G}\}$$

is linearly independent in $L^2(G)$.

We remark that this statement is quite easy to prove in the case when G is cyclic.

The second claim is used to establish the relations (1.1) assuming the hypotheses of Theorem 1.3.1. Suppose G is a finite abelian group and let $H = \mathrm{Hom}_{\mathbf{Z}}(G, \mathbf{R}/\mathbf{Z})$. Consider the space $L^2_{\mathrm{odd}}(H)$ consisting of functions $f : H \to \mathbf{C}$ which satisfy $f(-\phi) = -f(\phi)$ for all $\phi \in H$. For each $g \in G$, define the function $S_g \in L^2(H)$ by

$$S_g(\phi) = B_1(\phi(g)) \qquad \text{for all } \phi \in H.$$

Note that these functions lie in $L^2_{\mathrm{odd}}(H)$ since B_1 is an odd function. Crucially, however, much more is true:

Theorem 1.3.8 (cf. [6, Proposition 1.2])**.** *The space* $L^2_{\mathrm{odd}}(H)$ *is spanned by the functions* S_g *for* $g \in G$.

We remark that these functions are closely related to the Stickelberger distribution associated with B_1 described in [17, Chapter 2]. As in [6, 9], the proof of this theorem relies on Dirichlet's theorem that $L(1, \chi) \neq 0$ for a nontrivial Dirichlet character χ where $L(s, \chi)$ denotes the Dirichlet L-function associated with χ.

Proof of Theorem 1.3.1

We make some preliminary observations before stating the proof. Let $H = \mathrm{Hom}_{\mathbf{Z}}\left(\Lambda/\mathbf{Z}^n, \mathbf{R}/\mathbf{Z}\right)$. Given $u = (u_1, \ldots, u_n) \in \mathbf{R}^n$, define $h_u \in L^2(H)$ to be the function

$$h_u(\phi) = \left(\sum_{\substack{i=1\\ \pi_i=\phi}}^{n} u_i\right) - \left(\sum_{\substack{i=1\\ \pi_i=-\phi}}^{n} u_i\right) \quad \text{for all } \phi \in H.$$

Also define as above, for each $\lambda \in \Lambda \cap [0,1)^n$, the function $S_\lambda : H \to \mathbf{C}$:

$$S_\lambda(\phi) = B_1(\phi(\lambda + \mathbf{Z}^n)) = \begin{cases} \{\phi(\lambda + \mathbf{Z}^n)\} - 1/2, & \phi(\lambda + \mathbf{Z}^n) \neq 0 + \mathbf{Z} \\ 0, & \phi(\lambda + \mathbf{Z}^n) = 0 + \mathbf{Z} \end{cases}$$

The most important property about these functions is that they are odd functions; we have $S_\lambda(-\phi) = -S_\lambda(\phi)$ for each $\phi \in H$ and $\lambda \in \Lambda \cap [0,1)^n$.

Observe that for any $\lambda \in \Lambda \cap [0,1)^n$, we have

$$\sum_{i=1}^{n} u_i \lambda_i = \left(\sum_{i=1}^{n} u_i S_\lambda(\pi_i)\right) + \frac{1}{2}\left(\sum_{\substack{i=1\\ \lambda_i \neq 0}}^{n} u_i\right).$$

Since $\phi \mapsto -\phi$ is a permutation of H, we may write the first term as

$$\begin{aligned}
\sum_{\phi \in H} \sum_{\substack{i=1\\ \pi_i=\phi}}^{n} u_i S_\lambda(\phi) &= \frac{1}{2} \sum_{\phi \in H} \left(\left(\sum_{\substack{i=1\\ \pi_i=\phi}}^{n} u_i S_\lambda(\phi)\right) + \left(\sum_{\substack{i=1\\ \pi_i=-\phi}}^{n} u_i S_\lambda(-\phi)\right)\right) \\
&= \frac{1}{2} \sum_{\phi \in H} \left(\left(\sum_{\substack{i=1\\ \pi_i=\phi}}^{n} u_i\right) - \left(\sum_{\substack{i=1\\ \pi_i=-\phi}}^{n} u_i\right)\right) S_\lambda(\phi) \\
&= \frac{|H|}{2} \langle h_u, S_\lambda \rangle
\end{aligned}$$

where the second-to-last equality follows from the fact that S_λ is an odd function. So we conclude that for any $u \in \mathbf{R}^n$ with corresponding $h_u \in L^2(H)$ as defined above, and for any $\lambda \in \Lambda \cap [0,1)^n$, we have

$$\sum_{i=1}^{n} u_i \lambda_i = \frac{1}{2} \left(|H| \langle h_u, S_\lambda \rangle + \sum_{\substack{i=1 \\ \lambda_i \neq 0}}^{n} u_i \right). \tag{1.4}$$

Proof of the if direction of Theorem 1.3.1. We start with the easier direction. Assume $u \in \mathbf{R}^n$ satisfy the relations (1.1) and (1.2) and let $\lambda \in \Lambda \cap [0,1)^n$. The relations (1.1) imply that h_u is the zero function, so by (1.4) we may therefore write

$$\sum_{i=1}^{n} u_i \lambda_i = \frac{1}{2} \sum_{\substack{i=1 \\ \lambda_i \neq 0}}^{n} u_i = \frac{1}{2} \sum_{K} \sum_{\substack{i=1 \\ \lambda_i \neq 0 \\ \ker \pi_i = K}}^{n} u_i = \frac{1}{2} \sum_{\substack{K \\ \lambda + \mathbf{Z}^n \notin K}} \sum_{\substack{i=1 \\ \ker \pi_i = K}}^{n} u_i$$

where the outer sums are over all subgroups $K \in \{\ker \pi_i : i = 1, 2, \ldots, n\}$. By (1.2), the inner sums of the double sum on the right always vanish, and therefore the whole expression equals zero. $\square$

Proof of the only if direction. Let $u \in \mathbf{R}^n$ with corresponding $h_u \in L^2(H)$ as defined above, and assume that $\langle u, \lambda \rangle = 0$ for every $\lambda \in \Lambda \cap [0,1)^n$. For every pair $\lambda, \mu \in \Lambda \cap [0,1)^n$ such that $\lambda + \mathbf{Z}^n = -\mu + \mathbf{Z}^n$, we have

$$\sum_{\substack{i=1 \\ \lambda_i \neq 0}}^{n} u_i = \left(\sum_{i=1}^{n} u_i \lambda_i \right) + \left(\sum_{i=1}^{n} u_i \mu_i \right) = 0 \tag{1.5}$$

by our assumption that both the terms in the middle vanish. Hence

$$\frac{|H|}{2} \langle h_u, S_\lambda \rangle = \sum_{i=1}^{n} u_i \lambda_i = 0$$

for every $\lambda \in \Lambda \cap [0,1)^n$ by equation (1.4). So by Theorem 1.3.8, h_u is orthogonal to every odd function in $L^2(H)$ and therefore must be an even function. But h_u is an odd function by definition. It follows h_u must be the zero map, and therefore the relations (1.1) hold.

We next show that the relations (1.2) hold as well. From (1.5), we have

$$\left(\sum_{i=1}^{n} u_i\right) - \sum_{\substack{i=1 \\ \lambda_i=0}}^{n} u_i = 0$$

for all $\lambda \in \Lambda \cap [0,1)^n$ which implies

$$\left(\sum_{i=1}^{n} u_i\right) \mathbf{1}_{\Lambda/\mathbf{Z}^n} - \sum_{i=1}^{n} u_i \mathbf{1}_{\ker \pi_i} = \mathbf{0}$$

where $\mathbf{1}_K \in L^2(\Lambda/\mathbf{Z}^n)$ denotes the indicator function of the subgroup K of $\Lambda/\mathbf{Z}^n$ and $\mathbf{0}$ denotes the zero map. We may rewrite this sum as

$$\left(\sum_{i=1}^{n} u_i\right) \mathbf{1}_{\Lambda/\mathbf{Z}^n} - \sum_{K} \left(\sum_{\substack{i=1 \\ \ker \pi_i = K}}^{n} u_i\right) \mathbf{1}_K = \mathbf{0}$$

where the second sum is over all subgroups $K \in \{\ker \pi_i : i = 1, 2, \ldots, n\}$.

Let $e : \mathbf{R}/\mathbf{Z} \to \mathbf{C}^\times$ be the map $x + \mathbf{Z} \mapsto \exp(2\pi i x)$. Then $e \circ \pi_i \in \widehat{\Lambda/\mathbf{Z}^n}$ and, moreover, $\ker \pi_i = \langle e \circ \pi_i \rangle^\perp$ for each $i = 1, 2, \ldots, n$. We also have $\Lambda/\mathbf{Z}^n = \langle \chi_0 \rangle^\perp$, where χ_0 denotes the identity of $\widehat{\Lambda/\mathbf{Z}^n}$. It follows that $(\ker \pi_i)^\perp$ for $i = 1, 2, \ldots, n$ and $(\Lambda/\mathbf{Z}^n)^\perp$ are all cyclic subgroups of $\widehat{\Lambda/\mathbf{Z}^n}$. By Lemma 1.3.7, then, the set of indicator functions in the above linear combination are linearly independent. We conclude each of the coefficients of the indicator functions above are zero, and therefore the relations (1.2) hold. Note that there is no $\mathbf{1}_{\Lambda/\mathbf{Z}^n}$ term among the sum of $\mathbf{1}_K$'s due to the assumption that $\ker \pi_i \neq \Lambda/\mathbf{Z}^n$ for every i. $\square$

Proof of Lemma 1.3.7

Let G be a finite abelian group. If $f \in L^2(G)$, we define the Fourier transform $\hat{f} \in L^2(\widehat{G})$ by

$$\hat{f}(\chi) = \langle f, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) \overline{\chi(g)}$$

for every $\chi \in \widehat{G}$. Since the characters of G form an orthonormal basis of $L^2(G)$, we have in particular that

$$\widehat{\psi}(\chi) = \langle \psi, \chi \rangle = \begin{cases} 1, & \psi = \chi \\ 0, & \psi \neq \chi \end{cases} \tag{1.6}$$

for every $\psi, \chi \in \widehat{G}$.

Lemma 1.3.7 is essentially a consequence of the Poisson summation formula for finite abelian groups, stated below. We refer the reader to [14, Exercise 4.6] or [18, Chapter 12] for an exposition of this statement, noting the slight difference in presentation resulting from the $1/|G|$ factor in our definition of the inner product of $L^2(G)$.

Proposition 1.5.1 (Poisson summation formula)**.** *Let G be a finite abelian group, let $f \in L^2(G)$, and let K be a subgroup of G. Then*

$$\frac{1}{|G|} \sum_{k \in K} f(k) = \frac{1}{|K^{\perp}|} \sum_{\chi \in K^{\perp}} \hat{f}(\chi).$$

Lemma 1.5.2. *Let $\mathcal{K}$ be a collection of subgroups of G with the property that $\{\mathbf{1}_{K^{\perp}} : K \in \mathcal{K}\}$ is linearly independent in $L^2(\widehat{G})$. Then $\{\mathbf{1}_K : K \in \mathcal{K}\}$ is linearly independent in $L^2(G)$.*

Proof. Let $\mathcal{K}$ be such a collection, and suppose

$$\sum_{K \in \mathcal{K}} \alpha_K \mathbf{1}_K = 0$$

for some complex numbers α_K for $K \in \mathcal{K}$. Thus for any character $\psi \in \widehat{G}$, we obtain

$$\sum_{K \in \mathcal{K}} \frac{\alpha_K}{|K^\perp|} \sum_{\chi \in K^\perp} \widehat{\psi}\,(\chi) = \sum_{K \in \mathcal{K}} \frac{\alpha_K}{|G|} \sum_{k \in K} \psi\,(k) = \left\langle \sum_{K \in \mathcal{K}} \alpha_K \mathbf{1}_K, \overline{\psi} \right\rangle = 0$$

by the Poisson summation formula. On the other hand, by (1.6), the left hand side simplifies to

$$\sum_{\substack{K \in \mathcal{K} \\ \psi \in K^\perp}} \frac{\alpha_K}{|K^\perp|} = \sum_{K \in \mathcal{K}} \frac{\alpha_K}{|K^\perp|} \mathbf{1}_{K^\perp}(\psi).$$

It follows that the linear combination of functions

$$\sum_{K \in \mathcal{K}} \frac{\alpha_K}{|K^\perp|} \mathbf{1}_{K^\perp} \in L^2(\widehat{G})$$

is the zero function. Since the functions $\{\mathbf{1}_{K^\perp} : K \in \mathcal{K}\}$ are assumed to be linearly independent, we get that each $\alpha_K = 0$ which is what we wanted to show. □

Recall Lemma 1.3.7, which claims $\{\mathbf{1}_K \ : \ K^\perp$ is a cyclic subgroup of $\widehat{G}\}$ is linearly independent in $L^2(G)$.

Proof of Lemma 1.3.7. Let $\mathcal{K} = \{\langle \chi \rangle^\perp : \chi \in \widehat{G}\}$. By the preceding lemma, it suffices to show that set of functions

$$\{\mathbf{1}_{K^\perp} : K \in \mathcal{K}\} = \{\mathbf{1}_{\langle \chi \rangle} : \chi \in \widehat{G}\}$$

is linearly independent in $L^2(\widehat{G})$.

The cyclic subgroups of $\widehat{G}$ form a partially ordered set with respect to inclusion. Hence, by taking any linear extension of this poset, we enumerate these subgroups as

$\langle \chi_1 \rangle, \langle \chi_2 \rangle, \ldots, \langle \chi_n \rangle$ in such a way that $i < j$ implies there is an element of $\langle \chi_j \rangle$ not in $\langle \chi_i \rangle$. This implies that the matrix

$$A_{i,j} = \begin{cases} 1, & \chi_i \in \langle \chi_j \rangle \\ 0, & \text{otherwise}, \end{cases}$$

where $1 \leq i, j \leq n$, is upper triangular with ones along the diagonal. It follows that the functions in $\{\mathbf{1}_{\langle \chi \rangle} : \chi \in \widehat{G}\}$ are linearly independent, as they are linearly independent when restricted to $\{\chi_1, \ldots, \chi_n\}$. $\square$

Proof of Theorem 1.3.8

Let G be a finite abelian group written additively, and let $H = \mathrm{Hom}(G, \mathbf{R}/\mathbf{Z})$. We wish to show that the space $L^2_{\mathrm{odd}}(H)$ of odd functions $f : H \to \mathbf{C}$ is spanned by the functions $S_g : H \to \mathbf{C}$ defined by $S_g(\phi) = B_1(\phi(g))$ for each $g \in G$. The proof of this statement is outlined in this section, and follows the methods of [9, 6] by explicitly finding $\dim(L^2_{\mathrm{odd}}(H))$ many linearly independent vectors in $\mathrm{span}(S_g : g \in G)$.

Some preliminaries

By the structure theorem for finitely generated abelian groups, $G \simeq H$ is isomorphic to an additive group of the form

$$\bigoplus_{i=1}^{m} \mathbf{Z}/r_i$$

where $r_1, r_2, \ldots, r_m$ are positive integers such that $m \geq 1$ and $r_1 \mid r_2 \mid \cdots \mid r_m$. Now fix a minimal set of generators $\{g_1, \ldots, g_m\}$ of G, so that every element $g \in G$ can be written uniquely as $a_1 g_1 + \cdots + a_m g_m$ for some integers $a_1, \ldots, a_m$ satisfying

$0 \leq a_i < r_i$ for $i = 1, 2, \ldots, m$. Then the maps $\phi_i \in H$ for $i = 1, 2, \ldots, m$ defined by

$$\phi_i(g_j) = \begin{cases} 0 + \mathbf{Z} & i \neq j \\ \frac{1}{r_i} + \mathbf{Z} & i = j \end{cases}$$

are a minimal generating set for H in that every $\phi \in H$ can be written uniquely as $c_1\phi_1 + \cdots + c_m\phi_m$ for some integers $c_1, \ldots, c_m$ satisfying $0 \leq c_i < r_i$ for $i = 1, 2, \ldots, m$. Moreover, given $g \in G$ and $\phi \in H$, if $g = a_1 g_1 + \cdots + a_m g_m$ and $\phi = c_1\phi_1 + \cdots + c_m\phi_m$, then

$$\phi(g) = \frac{a_1 c_1}{r_1} + \cdots + \frac{a_m c_m}{r_m} + \mathbf{Z}.$$

Now let R denote the ring $\mathbf{Z}/r_1 \oplus \cdots \oplus \mathbf{Z}/r_m$ with componentwise multiplication, so that the additive group of R is isomorphic to G. For each $a = (a_1, \ldots, a_m) \in R$, define the function $S_a : R \to \mathbf{C}$ by

$$S_a(c) = B_1\left(\frac{a_1 c_1}{r_1} + \cdots + \frac{a_m c_m}{r_m}\right)$$

for each $c = (c_1, \ldots, c_m) \in R$.

As before, let $L^2_{\text{odd}}(R)$ denote the space of functions $f : R \to \mathbf{C}$ satisfying $f(-a) = -f(a)$ for all $a \in R$. Theorem 1.3.8, then, is established by proving the following proposition:

Proposition 1.6.1. *The functions in $\{S_a : a \in R\}$ span $L^2_{\text{odd}}(R)$.*

Before proceeding with the proof of Proposition 1.6.1, we review the notion of Dirichlet characters and establish the notation to be used in the remainder of this section. A reference can be found in [13, Section 9.1].

Dirichlet characters

Let $G = (\mathbf{Z}/r)^{\times}$ for some positive integer r. Then each character $\chi : G \to \mathbf{C}^{\times}$ extends to a completely multiplicative function $\chi : \mathbf{Z} \to \mathbf{C}$ by setting

$$\chi(n) := \begin{cases} \chi(n + r\mathbf{Z}), & \gcd(n, r) = 1 \\ 0, & \text{otherwise} \end{cases}$$

for each integer n. A function $\chi : \mathbf{Z} \to \mathbf{C}$ is called a *Dirichlet character* if it is constructed in this manner for some $r \geq 1$ and some $\chi \in \widehat{(\mathbf{Z}/r)^{\times}}$. The number r is called the *modulus* of χ. We define an equivalence relation $\sim$ on Dirichlet characters by declaring $\chi_1 \sim \chi_2$ if and only if they agree on their mutual support. A Dirichlet character χ is called *primitive* if the support of χ contains the support of every other Dirichlet character in the equivalence class $[\chi]$. Given a Dirichlet character χ, there exists a unique primitive Dirichlet character in the equivalence class $[\chi]$ and it is denoted χ^*. A primitive character χ^* is said to *induce* a Dirichlet character ψ if $\psi \in [\chi^*]$. If χ is a Dirichlet character, then the modulus of χ^* is called the *conductor* of χ.

Notation

We outline the notation used in the remainder of this section.

Arithmetic functions

Let $\mathbf{N}$ denote the positive integers.

- $\nu_p : \mathbf{N} \to \mathbf{Z}$ denotes the p-adic valuation: $\nu_p(k)$ is the largest exponent α such that $p^{\alpha} \mid k$.

- $\mathrm{d} : \mathbf{N} \to \mathbf{N}$ counts the number of divisors of an integer: we have $\mathrm{d}(k) = \prod_p (\nu_p(k) + 1)$ for all $k \geq 1$ where the product is over all primes p.

- $\mu : \mathbf{N} \to \mathbf{Z}$ is the Möbius function.

- $\varphi : \mathbf{N} \to \mathbf{N}$ is the Euler-phi function.

- We write (k, k') for the greatest common divisor of k and k'.

For $a = (a_1, \ldots, a_m) \in \mathbf{N}^m$, we also define

- $\mathrm{d}(a) := \mathrm{d}(a_1)\mathrm{d}(a_2) \cdots \mathrm{d}(a_m)$.

- $\mu(a) := \mu(a_1)\mu(a_2) \cdots \mu(a_m)$.

- $\varphi(a) := \varphi(a_1)\varphi(a_2) \cdots \varphi(a_m)$.

Dirichlet characters

Let $R = \oplus_{i=1}^m \mathbf{Z}/r_i$ be the ring defined above. The multiplicative group of units of R is given by

$$R^\times = \bigoplus_{i=1}^m (\mathbf{Z}/r_i)^\times .$$

Let $\widehat{R^\times}$ denote the group of characters of $R^\times$. Each $\chi \in \widehat{R^\times}$ corresponds uniquely to a tuple $(\chi_1, \ldots, \chi_m)$ for which $\chi_i \in \widehat{(\mathbf{Z}/r_i)^\times}$ for $i = 1, 2, \ldots, m$, and

$$\chi(a) = \chi_1(a_1) \cdots \chi_m(a_m)$$

for each $a = (a_1, \ldots, a_m) \in R^\times$. For a character $\chi_i : (\mathbf{Z}/r_i)^\times \to \mathbf{C}^\times$, we denote the corresponding Dirichlet character by $\chi_i : \mathbf{Z} \to \mathbf{C}$. For $\chi = (\chi_1, \ldots, \chi_m) \in \widehat{R^\times}$, we define

- $\chi\ : \mathbf{Z}^m \to \mathbf{C}$ by $\chi\ (a_1, \ldots, a_m) = \chi_1(a_1)\chi_2(a_2) \cdots \chi_m(a_m)$.

- $\chi^* : \mathbf{Z}^m \to \mathbf{C}$ by $\chi^*(a_1, \ldots, a_m) = \chi_1^*(a_1)\chi_2^*(a_2) \cdots \chi_m^*(a_m)$.

Here we are denoting by $\chi_i^* : \mathbf{Z} \to \mathbf{C}$ the primitive Dirichlet character inducing $\chi_i : \mathbf{Z} \to \mathbf{C}$.

Parameters associated with R.

For the ring R defined above, and for each $\chi = (\chi_1, \dots \chi_m) \in \widehat{R^\times}$, we define

- $r := (r_1, \dots, r_m)$.
- $f_\chi := (f_{\chi_1}, \dots, f_{\chi_m})$ where f_{χ_i} is the conductor of $\chi_i : \mathbf{Z} \to \mathbf{C}$.
- $q_\chi := (r_1/f_{\chi_1}, \dots, r_m/f_{\chi_m})$.

Everything else.

For two tuples of integers $a = (a_1, \dots, a_m), c = (c_1, \dots, c_m) \in \mathbf{Z}^m$, we write ac to denote the componentwise product $(a_1c_1, \dots, a_mc_m)$. If a and c have positive components, then we write $a \mid c$ and say a *divides* c if $a_i \mid c_i$ for all $i = 1, 2, \dots, m$. If a divides c, then we let $c/a := (c_1/a_1, \dots, c_m/a_m)$. Thus, for instance, $q_\chi = r/f_\chi$ where r, q_χ, f_χ are as above.

If $g, h : \mathbf{N}^m \to \mathbf{C}$, then let $*$ denote Dirichlet convolution over $\mathbf{N}^m$:

$$(g * h)(a) := \sum_{d \mid a} g(d)\, h(a/d) \quad \text{for all } a \in \mathbf{N}^m.$$

A decomposition of R

The group of units $R^\times$ of the ring R acts on $L^2(R)$ as follows: for a given $f \in L^2(R)$, $c \in R^\times$, the function $c \cdot f \in L^2(R)$ is defined so that

$$(c \cdot f)(a) = f(ca)$$

for each $a \in R$. An eigenvalue, eigenfunction pair (χ, w) of the action consists of a function $\chi : R^\times \to \mathbf{C}$ and a nonzero function $w \in L^2(R)$ such that for every $c \in R^\times$,

$$c \cdot w = \chi(c)\, w.$$

The vector space $L^2(R)$ can be decomposed into a direct sum

$$L^2(R) = \bigoplus_{\chi \in \widehat{R^\times}} \varepsilon_\chi \tag{1.7}$$

where, for each $\chi \in \widehat{R^\times}$, we denote the subspace of eigenfunctions corresponding to χ by ε_χ.

Proposition 1.6.2. *Let* $\chi \in \widehat{R^\times}$. *Then*

$$\varepsilon_\chi = \left\{ \frac{1}{|R^\times|} \sum_{b \in R^\times} \overline{\chi}(b)(b \cdot w) : w \in L^2(R) \right\}.$$

Proof. If w is an eigenfunction with eigenvalue χ, then w equals the average of $\overline{\chi}(b)(b \cdot w)$ over all $b \in R^\times$. Conversely, if $w \in L^2(R)$, then

$$\begin{aligned} c \cdot \sum_{b \in R^\times} \overline{\chi}(b)(b \cdot w) = \sum_{b \in R^\times} \overline{\chi}(b)(cb \cdot w) &= \chi(c) \sum_{b \in R^\times} \overline{\chi}(cb)(cb \cdot w) \\ &= \chi(c) \sum_{b \in R^\times} \overline{\chi}(b)(b \cdot w). \end{aligned}$$

□

We say that a character $\chi \in \widehat{R^\times}$ is *even* if $\chi(-1, \ldots, -1) = 1$ and *odd* if $\chi(-1, \ldots, -1) = -1$; note that these are the only two possible values for $\chi(-1, \ldots, -1)$ since

$$(\chi(-1, \ldots, -1))^2 = \chi((-1, \ldots, -1)^2) = \chi(1, \ldots, 1) = 1.$$

Observe that the functions in ε_χ are odd if and only if χ is odd.

For a given $\chi \in \widehat{R^\times}$ and $a' \in R$, let

$$w_{\chi, a'} := \sum_{b \in R^\times} \chi(b) S_{a'b} \in \varepsilon_\chi.$$

If $a \in \mathbf{Z}^m$ and $a' \in R$ is the image of a under the canonical map $\mathbf{Z}^m \to R$, then we also define $w_{\chi,a} := w_{\chi,a'}$.

The next theorem, proved by Reid in [9] for the case when $m = 1$, finds a basis of ε_χ in terms of these $w_{\chi,a}$ when χ is odd.

Proposition 1.6.3 (cf. [9, Theorem 5.13]). *For each odd character $\chi \in \widehat{R^\times}$, there are $\mathrm{d}(q_\chi)$ functions in $\{w_{\chi,a} : a \in \mathbf{N}^m, a \mid q_\chi\}$ and they are linearly independent.*

With this proposition, we can prove Proposition 1.6.1 and hence Theorem 1.3.8.

Proof of Proposition 1.6.1. For a tuple $f = (f_1, \ldots, f_m) \in \mathbf{N}^m$, let $\hat{\varphi}_{\text{odd}}(f)$ denote the number of odd characters $\chi \in \widehat{R^\times}$ such that $f = f_\chi$. Using Proposition 1.6.3 and the decomposition (1.7), we take the union of the sets $\{w_{\chi,a} : a \mid q_\chi\}$ over all odd characters χ to obtain $(\hat{\varphi}_{\text{odd}} * \mathrm{d})(r)$ linearly independent functions in $L^2_{\text{odd}}(R)$. We would therefore like to show that this number is equal to $\dim(L^2_{\text{odd}}(R))$.

Since $*$ is associative, we get

$$(\hat{\varphi}_{\text{odd}} * \mathrm{d})(r) = (\hat{\varphi}_{\text{odd}} * 1 * 1)(r) = \sum_{f \mid r} (\hat{\varphi}_{\text{odd}} * 1)(f).$$

Now each term $(\hat{\varphi}_{\text{odd}} * 1)(f)$ in the sum is equal to the total number of odd characters of the group $G_f := \oplus_{i=1}^m (\mathbf{Z}/f_i)^\times$. This number is equal to zero if G_f is the trivial group, which is the case if and only if f_i equals one or two for every $i = 1, 2, \ldots, m$. Otherwise, $\{\psi \in \widehat{G_f} : \psi(-1, \ldots, -1) = 1\}$ is an order two subgroup of $\widehat{G_f}$ and hence there are $\frac{1}{2}\left|\widehat{G_f}\right| = \frac{1}{2}|G_f| = \frac{1}{2}\varphi(f)$ odd characters in $\widehat{G_f}$.

If we let $\delta(f) = 1$ whenever every component of f is either 1 or 2 and zero otherwise, then we obtain

$$(\hat{\varphi}_{\text{odd}} * \mathrm{d})(r) = \frac{1}{2} \sum_{f \mid r} (\varphi(f) - \delta(f)) = \frac{1}{2}(r_1 \cdots r_m - 2^s)$$

where s equals the number of $i \in \{1, 2, \ldots, m\}$ such that r_i is even. Hence we obtain that the dimension of $\operatorname{span}(S_a : a \in R)$ is at least $\frac{1}{2}(|R| - 2^s)$.

It remains to show that $\dim\left(L^2_{\text{odd}}(R)\right) = \frac{1}{2}\left(|R| - 2^s\right)$. Observe that the functions $\{\mathbf{1}_a - \mathbf{1}_{-a} : a \in R\}$ span $L^2_{\text{odd}}(R)$, where $\mathbf{1}_a \in L^2(R)$ denotes the indicator function of the element $a \in R$. Indeed, for any $h \in L^2_{\text{odd}}(R)$ we have

$$h = \frac{1}{2} \sum_{a \in R} h(a) \left(\mathbf{1}_a - \mathbf{1}_{-a}\right).$$

The dimension of $\operatorname{span}\left(\mathbf{1}_a - \mathbf{1}_{-a} : a \in R\right)$ is equal to one-half the number of elements $a \in R$ such that $a \neq -a$. But the elements $a \in R$ for which $a = -a$ are precisely the elements $(\epsilon_1 r_1/2, \ldots, \epsilon_m r_m/2) \in R$ where each $\epsilon_i = 0$ or 1 but $\epsilon_i = 0$ for all i such that r_i is odd. That is to say, the number of elements $a \in R$ such that $a = -a$ is exactly 2^s. We therefore conclude

$$\begin{aligned}
\dim L^2_{\text{odd}}(R) &= \dim \operatorname{span}\left(\mathbf{1}_a - \mathbf{1}_{-a} : a \in R\right) \\
&= \tfrac{1}{2}\left(|R| - 2^s\right) \\
&\leq \dim \operatorname{span}(S_a : a \in R) \\
&\leq \dim L^2_{\text{odd}}(R).
\end{aligned}$$

and hence equality holds throughout. Since $S_a \in L^2_{\text{odd}}(R)$ for each $a \in R$, we conclude that $L^2_{\text{odd}}(R) = \operatorname{span}(S_a : a \in R)$. □

Finding a basis for each eigenspace

It therefore remains to prove Proposition 1.6.3. For the rest of the paper, we fix some odd $\chi \in \widehat{R^\times}$ and let $q := (q_1, \ldots, q_m) := q_\chi$ and $f := (f_1, \ldots, f_m) := f_\chi$.

We start by finding an alternate representation for $w_{\chi,a}(c)$ given $a, c \in R$. This representation is based on [13, Theorem 9.9], which expresses the generalized Bernoulli number $B_{1,\chi}$ in terms of the Dirichlet L-function $L(s, \chi)$ evaluated at $s = 1$.

Proposition 1.6.4. *Let $a = (a_1, \ldots, a_m), c = (c_1, \ldots, c_m) \in R$. Then*

$$w_{\chi,a}(c) = \frac{i}{\pi} \sum_{k \geq 1} \frac{1}{k} \prod_{i=1}^{m} \left(\overline{\chi}_i^* \left(\frac{k a_i c_i}{(r_i, k a_i c_i)} \right) F_{\chi_i}((r_i, k a_i c_i)) \right)$$

where $F_{\chi_i}(\beta) = 0$ if β does not divide q_i, and otherwise

$$F_{\chi_i}(\beta) = \chi_i^* \left(\frac{q_i}{\beta} \right) \mu \left(\frac{q_i}{\beta} \right) \frac{\varphi(r_i) \tau(\chi_i^*)}{\varphi(r_i/\beta)}. \tag{1.8}$$

The factor $\tau(\chi_i^*)$ above denotes the Gauss sum of the primitive character χ_i^*:

$$\tau(\chi_i^*) := \sum_{t \in (\mathbf{Z}/f_i)^\times} \chi_i^*(t)\, e(t/f_i)\,.$$

For our purposes, the only thing we need to know about this quantity is that it is nonzero [13, Theorem 9.7].

Proof. Consider the quantity

$$A := \sum_{\substack{b \in R^\times \\ \theta_{acb} \notin \mathbf{Z}}} \chi(b) \log(1 - e(\theta_{acb}))$$

where $e(x) := \exp(2\pi i x)$, the logarithm is the principal branch, and

$$\theta_{acb} := \frac{a_1 c_1 b_1}{r_1} + \cdots + \frac{a_m c_m b_m}{r_m}.$$

In the sum, we replace $\log(1 - e(\theta_{acb}))$ with its real and imaginary parts:

$$\log(1 - e(\theta_{acb})) = \log|2\sin(\pi\theta_{acb})| + i\pi(\{\theta_{acb}\} - 1/2)\,,$$

then distribute to obtain two sums. The first of these is zero which can be seen by noting that $|\sin(\pi\theta_{acb})| = |\sin(\pi\theta_{-acb})|$ and therefore we can replace each $\chi(b)$ with

$\frac{1}{2}(\chi(b)+\chi(-b))$ which is zero since χ is odd. The second sum is therefore equal to A, and from it we recover $w_{\chi,a}(c)$:

$$A = i\pi \sum_{\substack{b\in R^\times \\ \theta_{acb}\notin \mathbf{Z}}} \chi(b)(\{\theta_{acb}\} - 1/2) = i\pi w_{\chi,a}(c).$$

On the other hand, we use the Taylor expansion of the logarithm to obtain

$$A = \sum_{\substack{b\in R^\times \\ \theta_{acb}\notin \mathbf{Z}}} \chi(b) \sum_{k\geq 1} -\frac{e(k\theta_{acb})}{k} = -\sum_{k\geq 1} \frac{1}{k} \sum_{b\in R^\times} \chi(b) e(k\theta_{acb}).$$

Since the double sum on the left is a finite sum of convergent series, we may interchange the sums. The second equality holds since, after interchanging, the terms of the inner sum for which $\theta_{acb} \in \mathbf{Z}$ sum to zero. Indeed, over such terms we may pull out $e\left(k\theta_{acb}\right) = 1$ and replace each $\chi(b)$ with $\frac{1}{2}(\chi(b)+\chi(-b))$ which is zero as before. We may therefore write the inner sum as the product

$$\prod_{i=1}^{m} \left(\sum_{b_i \in (\mathbf{Z}/r_i)^\times} \chi_i\left(b_i\right) e\left(\frac{k a_i c_i b_i}{r_i}\right)\right).$$

Now let $\beta_{i,k} := (r_i, k a_i c_i)$. Applying [13, Theorem 9.12], each factor above can be written

$$\overline{\chi_i}^* \left(\frac{k a_i c_i}{\beta_{i,k}}\right) \chi_i^* \left(\frac{q_i}{\beta_{i,k}}\right) \mu \left(\frac{q_i}{\beta_{i,k}}\right) \frac{\varphi\left(r_i\right)}{\varphi\left(r_i/\beta_{i,k}\right)} \tau\left(\chi_i^*\right)$$

if $\beta_{i,k} \mid q_i$. Otherwise it is zero. □

Following Reid in [9, Theorem 5.16], it is more convenient to prove Proposition 1.6.3 by showing that the functions

$$v_{\chi,a} := \sum_{d\mid a} \mu\left(d\right) \overline{\chi}^*\left(d\right) w_{\chi,a/d}$$

over all $a \in \mathbf{N}^m$ which divide q are linearly independent in $L^2(R)$. We can accomplish this by showing that the matrix

$$(v_{\chi,a}(c))_{a,c}$$

is nonsingular, where the rows and columns of the matrix are indexed by tuples $a, c \in \mathbf{N}^m$ such that $a \mid q$ and $c \mid q$, and $v_{\chi,a}(c) := v_{\chi,a}(c')$ where c' is the image of c under the canonical map $\mathbf{Z}^m \to R$. This is done over the next three propositions. Proposition 1.6.5 finds an ordering of the divisors of q so that:

1. the indices (a, c) of the antidiagonal entries of the matrix satisfy $ac = q$.

2. The indices (a, c) to the right of the antidiagonal entries satisfy $ac \nmid q$.

Proposition 1.6.6 shows that $v_{\chi,a}(c) = 0$ for all $a \mid q$ and $c \mid q$ satisfying $ac \nmid q$. Finally, this paper concludes with Proposition 1.6.7, which shows that $v_{\chi,a}(c) \neq 0$ for all $a, c \in \mathbf{N}^m$ satisfying $ac = q$ and hence the matrix is indeed nonsingular.

Proposition 1.6.5. *There exists a linear ordering*

$$a^{(1)} < a^{(2)} < \cdots < a^{(N)}$$

of tuples in $\mathbf{N}^m$ *which divide* q*, so that:*

1. For all $i, j = 1, 2, \ldots, N$, $i < j$ implies $a^{(j)} \nmid a^{(i)}$.

2. For all $i = 1, 2, \ldots, N$, $a^{(i)} a^{(N-i+1)} = q$.

Proof. The tuples in $\mathbf{N}^m$ which divide q form a graded poset with rank function given by $\operatorname{rank}(a_1, \ldots, a_m) = \sum_{i=1}^m \sum_p \nu_p(a_i)$ where the inner sum is over all primes p. To construct our ordering, we first specify that $a < b$ whenever $\operatorname{rank}(a) < \operatorname{rank}(b)$. Then, we arbitrarily order the elements within each level set $\operatorname{rank}^{-1}(j)$ for each j in the range $0 \leq j < \operatorname{rank}(q)/2$. If $\operatorname{rank}(q)$ is even, we further take the elements a with rank equal to $\operatorname{rank}(q)/2$ which do not satisfy $a^2 = q$, group them into pairs of the form $(a, q/a)$, choose a unique representative from each such pair, and arbitrarily order these representatives. Next, we set $q/a > q/b$ whenever $a < b$ and $\operatorname{rank}(a) = \operatorname{rank}(b) \leq \operatorname{rank}(q)/2$. Finally, we set $a^{((N+1)/2)} = a$ if there exists a which satisfies $a^2 = q$. The result is a linear ordering satisfying (1) and (2). □

Proposition 1.6.6 (cf. [9, Proposition 5.17(i)], [19, Lemma 4.18]). *If $a, c \in \mathbf{N}^m$ divide q but $ac \nmid q$, then $v_{\chi,a}(c) = 0$.*

Proof. Assume $a \mid q$ and $c \mid q$ but $ac \nmid q$. Then there exists some $i \in \{1, 2, \ldots, m\}$, $\alpha \geq 0$, and prime p_i such that $p_i^{\alpha+1} \mid a_i c_i$ and $p_i^{\alpha} \mid q_i$ but $p_i^{\alpha+1} \nmid q_i$. The key insight (taken from the above two references) is that are two different possible reasons why $v_{\chi,a}(c)$ must equal zero, depending on whether or not $p_i \mid f_i$.

First suppose $p_i \mid f_i$. Let $d \mid a$ and assume that d_i is coprime to f_i. Then p_i does not divide d_i and therefore $p_i^{\alpha+1}$ divides $a_i c_i / d_i$. We also have $p_i^{\alpha} \mid q_i$ and $p_i \mid f_i$ which means $p_i^{\alpha+1} \mid r_i$. It follows that $p_i^{\alpha+1}$ divides $(r_i, a_i c_i/d_i)$ and hence $p_i^{\alpha+1} \mid (r_i, k a_i c_i/d_i)$ for every $k \geq 1$. Since $p_i^{\alpha+1}$ does not divide q_i, it follows that $(r_i, k a_i c_i/d_i)$ does not divide q_i for any $k \geq 1$. By Proposition 1.6.4, then, we conclude $w_{\chi,a/d}(c) = 0$ for every $d \mid a$ such that d_i is coprime to f_i. But the only terms in the sum

$$v_{\chi,a}(c) = \sum_{d \mid a} \mu(d)\, \overline{\chi}^*(d)\, w_{\chi,a/d}(c)$$

which can be nonzero are the ones for which d is coprime to f in *every* component, including component i. This is due to the presence of the $\overline{\chi}^*(d)$ term which vanishes if this is not the case. It follows that $v_{\chi,a}(c) = 0$ in the case $p_i \mid f_i$.

Now suppose $p_i \nmid f_i$. Since $p_i^{\alpha+1} \mid a_i c_i$, it follows that p_i must divide both a_i and c_i since both a_i and c_i are divisors of q_i and $p_i^{\alpha+1} \nmid q_i$. In particular, p_i must divide a_i. Now let $p := (1, \ldots, 1, p_i, 1, \ldots, 1)$ and let $p' = p^{\nu_{p_i}(a_i)}$ so that the i^{th} component of a/p' is not divisible by p_i. Because the presence of the $\mu(d)$ term ensures that the sum $v_{\chi,a}(c)$ is only over d with squarefree components, we can group the sum as follows:

$$\sum_{d \mid \frac{a}{p'}} \left(\mu\left(d\right)\overline{\chi}^*\left(d\right) w_{\chi,a/d}(c) + \mu\left(pd\right)\overline{\chi}^*\left(pd\right) w_{\chi,a/pd}(c)\right). \tag{1.9}$$

Since $\mu(pd) = -\mu(d)$ for every $d \mid \frac{a}{p'}$, it suffices to show

$$\overline{\chi}^*\left(d\right) w_{\chi,a/d}(c) = \overline{\chi}^*\left(pd\right) w_{\chi,a/pd}(c)$$

for every $d \mid \frac{a}{p'}$ in order to establish $v_{\chi,a}(c) = 0$. By Proposition 1.6.4, it suffices to show that

$$\begin{aligned} &\overline{\chi}_i^*(d_i)\overline{\chi}_i^*\left(\frac{ka_ic_i/d_i}{(r_i, ka_ic_i/d_i)}\right) F_{\chi_i}((r_i, ka_ic_i/d_i)) \\ &= \overline{\chi}_i^*(p_id_i)\overline{\chi}_i^*\left(\frac{ka_ic_i/p_id_i}{(r_i, ka_ic_i/p_id_i)}\right) F_{\chi_i}((r_i, ka_ic_i/p_id_i)) \end{aligned}$$

for every $d \mid \frac{a}{p'}$ and every $k \geq 1$. But since $p_i \nmid f_i$ and $p_i \nmid d_i$, we have $p_i^{\alpha+1} \nmid r_i$ while $p_i^{\alpha+1} \mid ka_ic_i/d_i$. It follows that $(r_i, ka_ic_i/d_i) = (r_i, ka_ic_i/p_id_i)$, and hence the above equality indeed holds for all $d \mid \frac{a}{p'}$ and all $k \geq 1$. □

Proposition 1.6.7 (cf. [9, Proposition 5.17(ii)])**.** *Let $a, c \in \mathbf{N}^m$ be divisors of q such that $ac = q$. Then $v_{\chi,a}(c) \neq 0$.*

Proof. Suppose $d \mid a$ and each component d_i of d is squarefree and coprime to f_i. From Proposition 1.6.4 we can write

$$\mu(d)\overline{\chi}^*(d)w_{\chi,a/d}(c) = \frac{i}{\pi}\sum_{k\geq 1}\frac{1}{k}\prod_{i=1}^{m}\left(\mu(d_i)\overline{\chi}_i^*\left(\frac{kq_i}{\beta_{i,k}}\right)F_{\chi_i}(\beta_{i,k})\right)$$

where $\beta_{i,k} := (r_i, kq_i/d_i) = (r_i, k(a_i/d_i)c_i)$. Now consider the factor

$$\mu(d_i)\overline{\chi}_i^*\left(\frac{kq_i}{\beta_{i,k}}\right)F_{\chi_i}(\beta_{i,k}). \tag{1.10}$$

which appears in the above expression. We start by showing that, regardless of whether or not $\beta_{i,k}$ divides q_i, expression (1.10) simplifies to

$$\tau\left(\chi_i^*\right) \cdot \frac{\varphi\left(r_i\right)}{\varphi\left(d_i f_i\right)} \cdot \mu\left(\left(d_i, k\right)\right) \varphi\left(\left(d_i, k\right)\right) \overline{\chi_i}^*\left(k\right). \tag{1.11}$$

Observe that $\beta_{i,k}$ divides q_i if and only if the last equality of

$$\frac{q_i}{\beta_{i,k}} = \frac{q_i}{\left(r_i, kq_i/d_i\right)} = \frac{d_i}{\left(d_i f_i, k\right)} = \frac{d_i}{\left(d_i, k\right)} \tag{1.12}$$

holds, as d_i is coprime to f_i by assumption. Therefore, if $\beta_{i,k} \mid q_i$, then plugging in $d_i/(d_i,k)$ for $q_i/\beta_{k,i}$ in (1.10) quickly yields (1.11). On the other hand, if $\beta_{i,k} \nmid q_i$, then (1.10) also simplifies to (1.11). Indeed, in this case (1.10) just equals zero since $F_{\chi_i}(\beta_{k,i})$ is zero by definition. Since $\beta_{i,k} \nmid q_i$, the last equation in (1.12) fails to hold. This implies k shares a factor with f_i, and hence $\overline{\chi}_i^*(k) = 0$. So (1.11) is zero as well.

We therefore can write

$$\mu\left(d\right) \overline{\chi}^*\left(d\right) w_{\chi,a/d}(c) = \frac{C'_{a,\chi}}{\varphi\left(d\right)} \sum_{k \geq 1} \frac{\overline{\chi}\left(k\right) g_d\left(k\right)}{k}$$

where:

- $C'_{a,\chi}$ is a nonzero constant that depends only on a and χ
- $\overline{\chi} : \mathbf{Z} \to \mathbf{C}$ is an odd Dirichlet character defined by

  $$\overline{\chi}\left(k\right) := \prod_{i=1}^{m} \overline{\chi_i}^*\left(k\right)$$

  (Note: we do not put a star since this Dirichlet character may not be primitive).
- $g_d : \mathbf{Z} \to \mathbf{Z}$ is the function given by

  $$g_d\left(k\right) = \prod_{i=1}^{m} \mu\left(\left(d_i, k\right)\right) \varphi\left(\left(d_i, k\right)\right).$$

We now further simplify the right hand side above. Let $h_d : \mathbf{Z} \to \mathbf{C}$ be the function

$$h_d(k) = \overline{\chi}(k)(\mu * g_d)(k) = \overline{\chi}(k) \sum_{\ell | k} \mu(\ell) g_d(k/\ell),$$

where $*$ denotes Dirichlet convolution. For $k \geq 1$, $h_d(k)$ is zero unless k is square-free. Indeed, if p is a prime such that p^α is the highest power of p dividing k and $\alpha \geq 2$, then

$$h_d(k) = \overline{\chi}(k) \sum_{\ell | \frac{k}{p^\alpha}} \left(\mu(\ell) g_d\left(\frac{k}{\ell}\right) + \mu(p\ell) g_d\left(\frac{k}{p\ell}\right)\right),$$

and since $g_d(\ell)$ depends only on the square-free part of ℓ, the terms in each summand cancel each other out as in (1.9). Thus we may write

$$h_d(k) = \overline{\chi}(k) \sum_{\ell | k} \mu(k/\ell) g_d(\ell) = \overline{\chi}(k) \mu(k) \sum_{\ell | k} \mu(\ell) g_d(\ell).$$

If $n_d(p)$ denotes the number of indices $i \in \{1, 2, \ldots, m\}$ such that $p \mid d_i$, then

$$g_d(\ell) = \prod_{p | \ell} (1-p)^{n_d(p)}$$

and so

$$\begin{aligned} \sum_{k \geq 1} \frac{h_d(k)}{k} &= \sum_{k \geq 1} \frac{\overline{\chi}(k) \mu(k)}{k} \prod_{p | k} \left(1 - (1-p)^{n_d(p)}\right) \\ &= \prod_p \left(1 - \frac{\overline{\chi}(p)}{p} \left(1 - (1-p)^{n_d(p)}\right)\right) \end{aligned}$$

where the first product appearing above is over all primes p dividing k, and the second product is over all primes p. From the first equality we see that the series on the left converges absolutely (and is in fact finite) since only finitely many primes p satisfy $n_d(p) \geq 1$. It is a basic fact of number theory [13, Theorem 4.9] that the sum $L(1, \overline{\chi}) = \sum_{k \geq 1} \overline{\chi}(k)/k$ converges and is nonzero, and since $h_d = \overline{\chi}(\mu * g_d)$ we have

$\overline{\chi} g_d = \overline{\chi} * h_d$ and therefore

$$\sum_{k\geq 1} \frac{\overline{\chi}\left(k\right) g_d\left(k\right)}{k} = \left(\sum_{k\geq 1} \frac{\overline{\chi}\left(k\right)}{k}\right) \left(\sum_{k\geq 1} \frac{h_d\left(k\right)}{k}\right).$$

Moreover, since the components of d are squarefree and $\varphi(p) = p - 1$ for every prime p, we have

$$\varphi\left(d\right) = \prod_p \left(p-1\right)^{n_d(p)}$$

and therefore

$$\mu\left(d\right) \overline{\chi}^*\left(d\right) w_{\chi,a/d}(c) = C_{a,\chi} \prod_p \gamma\left(p, n_d(p)\right)$$

where $C_{a,\chi}$ is nonzero and depends only on a and χ and

$$\gamma\left(p,k\right) := \frac{1}{\left(p-1\right)^k} \left(1 - \frac{\overline{\chi}\left(p\right)}{p}\right) + \left(-1\right)^k \frac{\overline{\chi}\left(p\right)}{p}.$$

Now we find an expression for $v_{\chi,a}\left(c\right)$. We have

$$v_{\chi,a}\left(c\right) = C_{a,\chi} \sum_{d|a'} \prod_p \gamma\left(p, n_d(p)\right) = C_{a,\chi} \sum_t N\left(t\right) \prod_p \gamma\left(p, t_p\right),$$

where the sum on the right hand side is over all tuples of nonnegative integers $t = (t_2, t_3, t_5, \ldots)$ indexed by the primes, $a' = (a_1', \ldots, a_m')$ where a_i' is the largest squarefree divisor of a_i coprime to f_i for $i = 1, 2, \ldots, m$, and $N\left(t\right)$ counts the number of $d \mid a'$ such that $n_d(p) = t_p$ for all primes p. For a given tuple t, we have

$$N\left(t\right) = \prod_p \binom{n_{a'}(p)}{t_p},$$

thus

$$v_{\chi,a}\left(c\right) = C_{a,\chi} \sum_t \prod_p \binom{n_{a'}(p)}{t_p} \gamma\left(p, t_p\right) = C_{a,\chi} \prod_p \left(\sum_{k\geq 0} \binom{n_{a'}(p)}{k} \gamma\left(p,k\right)\right).$$

For a given prime p, by the binomial theorem, the inner sum is equal to 1 if $n_{a'}(p) = 0$, and otherwise equal to

$$\left(1 - \frac{\overline{\chi}(p)}{p}\right)\left(1 + \frac{1}{p-1}\right)^{n_{a'}(p)}.$$

So we conclude that

$$v_{\chi,a}(c) = C_{a,\chi} \prod_{\substack{p \\ n_{a'}(p) \geq 1}} \left(1 - \frac{\overline{\chi}(p)}{p}\right)\left(\frac{p}{p-1}\right)^{n_{a'}(p)} \neq 0.$$

$\square$

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Part II

Oriented matroids

CHAPTER 2

THE REAL BERGMAN FAN OF AN ORIENTED MATROID

Introduction

In tropical geometry one studies *tropical varieties*, or polyhedral complexes which are combinatorial counterparts to classical complex varieties. The most basic kind of tropical variety is a *tropical linear space*, which is the tropical analogue of a linear subspace of a complex vector space. In the simplest ("trivially-valued") case, all of the polyhedra in a tropical linear space are cones, and such a complex is also called a *Bergman fan*. Bergman fans are equivalent to matroids, in the sense that a Bergman fan canonically determines a matroid and vice-versa.

In the theory of oriented matroids, an important class of oriented matroids come from *zonotopes*. A zonotope is a polytope given by a Minkowski sum of line segments, and the face lattice of a zonotope is equivalent to the data of an oriented matroid [20]. While it may be the case that there are many zonotopes which determine the same oriented matroid, or none at all (the so-called "nonrealizable" oriented matroids), one might ask to what extent zonotopes play the role of Bergman fans in the context of oriented matroids, beyond simply "polyhedral representations of combinatorial objects."

It turns out that there is a close analogy between zonotopes and Bergman fans, and the aim of this paper is make it more explicit. We do this not by considering a zonotope directly, but rather a fan we call the *real Bergman fan* which projects onto the face fan of a zonotope in the realizable case. This fan can be defined for any oriented matroid, realizable or not, and shares many of the same features of a Bergman fan. For example, the Bergman fan of a matroid M on the ground set

E actually has multiple fan structures. One, called the *fine subdivision* in [21], is a geometric realization of the order complex of the lattice of flats of M. Another, which also appears in [21], is called the *coarse subdivision* of M and arises as a subfan of the normal fan of matroid base polytope of M:

$$P_M := \mathsf{conv}\left(\sum_{f \in B} \mathbf{e}_f : B \text{ is a basis of } M\right) \subseteq \mathbf{R}^E.$$

Analogously, the real Bergman fan of an oriented matroid $\mathcal{M}$ on the ground set E, which we denote by $\Sigma_{\mathcal{M}}$, also admits multiple fan structures. Our version of the fine subdivision of $\Sigma_{\mathcal{M}}$ is a geometric realization of the order complex of the poset of vectors of $\mathcal{M}$, while our version of the coarse subdivision of $\Sigma_{\mathcal{M}}$ is a subfan of the outer normal fan of the polytope

$$P_M^{\pm} := \mathsf{conv}\left(\sum_{f \in B} \pm\mathbf{e}_f : B \text{ is a basis of } M\right) \subseteq \mathbf{R}^E$$

where M is the underlying matroid of $\mathcal{M}$.

Our aim in this paper is to understand the cones in the coarse subdivision of $\Sigma_{\mathcal{M}}$. In the process we describe *all* the normal cones of $P_M^{\pm}$ for *any* loop-free matroid M, realizable or not. In some sense this work has been completed already to various degrees in the literature, for instance in the work of Ardila and Klivans on Bergman fans in [21]; the work of Ardila, Klivans, Reiner, and Williams on positive Bergman fans in [22, 23]; Fujishige's work on faces of submodular base polyhedra [24]; and Kim's work on flag enumerations of base polytopes [25]. The main contribution of this paper is to bring the ideas in these references together to give an intrinsic characterization of the faces of $P_M^{\pm}$, and use it to give an extreme ray description of the cones in the coarse subdivision of $\Sigma_{\mathcal{M}}$.

As an illustration of this work, we work out the case when $\mathcal{M}$ is uniform. Here, the coarse subdivision of $\Sigma_{\mathcal{M}}$ takes on a very simple form. It behaves like the face fan of a zonotope in the following sense: excluding the origin, the face lattice of the coarse subdivision of $\Sigma_{\mathcal{M}}$ is anti-isomorphic to the poset of nonzero vectors of $\mathcal{M}$.

Preliminaries

We assume the reader is familiar with the basic theory of matroids and oriented matroids; especially the notions of vectors and covectors of oriented matroids. See Oxley's text [26] for a standard reference on matroids, and [27] or [28] for a reference on oriented matroids. We state many of our results in terms of polytopes and normal fans of polytopes. For a reference on these topics, we refer the reader to Ziegler's text [29].

Throughout this paper, unless otherwise indicated, $\mathcal{M}$ will denote an oriented matroid of rank r on the ground set $E = \{1, 2, \ldots, n\}$. The vector space $\mathbf{R}^E$ has the canonical basis $\{\mathbf{e}_f : f \in E\}$. Given a vector $\omega \in \mathbf{R}^E$, the support of ω is denoted by $\mathsf{supp}(\omega)$. We reserve the letters X, Y, Z for signed subsets of E; that is, elements of $\{-1, 0, 1\}^E$. We follow the usual notation of [27] regarding signed sets. We reserve the letters F, R, S, T for ordinary subsets of E. For a signed subset X of E, we also write

$$\mathbf{e}_X := \left(\sum_{f \in X^+} \mathbf{e}_f\right) - \left(\sum_{f \in X^-} \mathbf{e}_f\right) \in \mathbf{R}^E$$

to emphasize the fact that X lives in $\mathbf{R}^E$.

Main results and examples

Statement of the main theorem

Let $\mathcal{M}$ be an oriented matroid. The main invariant associated to $\mathcal{M}$ that we consider in this paper is the polyhedral fan $\Sigma_{\mathcal{M}}$ defined as follows:

$$\Sigma_{\mathcal{M}} := \{\mathsf{cone}(\mathscr{F}) : \mathscr{F} \text{ is a flag of conformal vectors of } \mathcal{M}\}$$

where

$$\mathsf{cone}(\mathscr{F}) := \mathsf{cone}(\mathbf{e}_{X_1}, \mathbf{e}_{X_2}, \ldots, \mathbf{e}_{X_k})$$

for each flag $\mathscr{F} : X_1 < X_2 < \cdots < X_k$ of conformal vectors in $\mathcal{M}$. We call this fan the *fine subdivision* of $\Sigma_{\mathcal{M}}$. Note that $\mathcal{M}$ can be recovered from $\Sigma_{\mathcal{M}}$; one can see this by observing that the rays in this fan determine the vectors of $\mathcal{M}$, which in turn determine $\mathcal{M}$.

The main goal of this paper is to make explicit a very intricate combinatorial structure underlying this polyhedral complex, one which is not immediately apparent from this definition. To do this we require a few defintions.

Let $\mathcal{L}_N$ be the lattice of flats of a loop-free matroid N on the ground set S. Recall that a pair of flats F_1, F_2 in $\mathcal{L}_N$ form a *modular pair* if

$$\mathsf{rk}_N(F_1) + \mathsf{rk}_N(F_2) = \mathsf{rk}_N(F_1 \cup F_2) + \mathsf{rk}_N(F_1 \cap F_2).$$

Definition 2.3.1. A sublattice $\mathcal{D}$ of $\mathcal{L}_N$ is *initial* if $\emptyset, S \in \mathcal{D}$ and for all $F_1, F_2 \in \mathcal{L}_N$ we have

$$F_1, F_2 \in \mathcal{D} \qquad \Longleftrightarrow \qquad \begin{array}{l} F_1 \cup F_2 \in \mathcal{D} \\ F_1 \cap F_2 \in \mathcal{D} \\ F_1, F_2 \text{ form a modular pair.} \end{array}$$

Note that the above definition relies on the interpretation of the elements of $\mathcal{L}_N$ as subsets of the ground set S. Sublattices of this type have been studied in the more general context of submodular functions by Fujishige in [24, Section 3.3 (d)].

Definition 2.3.2. Let M be a matroid on the ground set E. Let $X \in \{-1, 0, 1\}^E$, and let S denote the support of X. Let $\mathcal{D}$ be an initial sublattice of the restriction

$M|S$. Define $\sigma(X, \mathcal{D}) \subseteq \mathbf{R}^E$ to be the cone with extreme ray description

$$\mathsf{cone}\left(\rho \in \mathbf{R}^E \ : \ \text{for some } F \in \mathcal{D}^\dagger, \quad \begin{matrix} \rho_e = X_e & \text{if } e \in F \\ \rho_e = \pm 1 & \text{if } e \in \mathsf{cl}_M(F) \smallsetminus F \\ \rho_e = 0 & \text{if } e \in E \smallsetminus \mathsf{cl}_M(F) \end{matrix} \right) \tag{2.1}$$

where $\mathcal{D}^\dagger$ is the set of nonempty $F \in \mathcal{D}$ such that $\mathsf{cl}_M(F)$ is connected in M.

By a *signed basis* of M, we mean a signed set $X \in \{-1, 0, 1\}^E$ whose support is equal to some basis of M.

Definition 2.3.3. Let M be a matroid on the ground set E. The *signed matroid polytope* $P_M^\pm$ of M is the polytope

$$P_M^\pm := \mathsf{conv}\left(\mathbf{e}_X : X \text{ is a signed basis of } M\right) \subseteq \mathbf{R}^E.$$

We are now ready to state our main theorem.

Theorem 2.3.4 (Main theorem)**.** *Let M denote the underlying matroid of $\mathcal{M}$. There is a subfan of the outer normal fan of $P_M^\pm$ whose support is exactly the support of $\Sigma_{\mathcal{M}}$. The cones $\sigma(X, \mathcal{D})$ in this fan are of the form (2.1) above, and are in bijection with pairs $(X, \mathcal{D})$ such that $X \in \{-1, 0, 1\}^E$ is a sign vector in which $X \cap F$ is a vector of $\mathcal{M}$ for each $F \in \mathcal{D}$, and $\mathcal{D}$ is an initial sublattice of $M|S$ where $S := \mathsf{supp}(X)$.* [1]

We call this fan structure the *coarse subdivision* of $\Sigma_{\mathcal{M}}$. When $\mathcal{M}$ is uniform, this fan structure behaves exactly like the face fan of a zonotope:

Corollary 2.3.5. *Suppose $\mathcal{M}$ is uniform. Then the poset (with respect to inclusion) of nonzero cones in the coarse subdivision of $\Sigma_{\mathcal{M}}$ is anti-isomorphic to the poset of nonzero vectors of $\mathcal{M}$.*

[1]The signed set $X \cap F$ satisfies $(X \cap F)^+ = X^+ \cap F$ and $(X \cap F)^- = X^- \cap F$.

Examples

We give three examples to illustrate the coarse subdivision of $\Sigma_{\mathcal{M}}$.

Example 2.3.6. Let $\mathcal{M}$ be the oriented matroid corresponding to the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

The underlying matroid M is $U_{1,3}$. The polytope $P_M^{\pm} \subseteq \mathbf{R}^3$ is an octahedron, and the maximal cones in the coarse subdivision of $\Sigma_{\mathcal{M}}$ consist of six two-dimensional cones. The polytope $P_{M^*}^{\pm}$ is a cuboctahedron, and $\Sigma_{\mathcal{M}^*}$ consists of two antiparallel rays. See Figure 2.1.

Example 2.3.7. Fix an orientation of the complete graph of K_4, and let $\mathcal{M}$ be the corresponding oriented matroid. Intersecting the coarse subdivision of $\Sigma_{\mathcal{M}}$ with the boundary of the 0-symmetric cube $[-1,1]^6$, we obtain a polyhedral complex that is linearly isomorphic to the subdivision of the boundary of the permutahedron shown in Figure 2.2.

Example 2.3.8. Let $\mathcal{M}$ be the oriented matroid dual to Ringel's nonrealizable uniform oriented matroid: $\mathcal{M}^* = \mathsf{Rin}(3,9)$. The intersection of the coarse subdivision of $\Sigma_{\mathcal{M}}$ with the boundary of $[-1,1]^9$ is shown in Figure 2.3.

Further remarks

The fan $\Sigma_{\mathcal{M}}$ is the oriented matroid analogue of the *Bergman fan* $\mathcal{B}(M)$ of a matroid M, originally defined by Sturmfels [30, Ch. 4]. The fine and coarse subdivisions of $\Sigma_{\mathcal{M}}$ parallel similar fan structures on the Bergman fan observed by Ardila and Klivans in [21]. If $\mathcal{M}$ is totally cyclic, so that $\mathcal{M}^*$ is acyclic, then the all ones vector $\mathbf{1} \in \mathbf{R}^E$ generates a ray in $\Sigma_{\mathcal{M}^*}$. The local fan structure of $\Sigma_{\mathcal{M}^*}$ around this ray coincides with the positive Bergman fan $\mathcal{B}^+(M)$ of Ardila, Klivans, Reiner, and Williams [23,

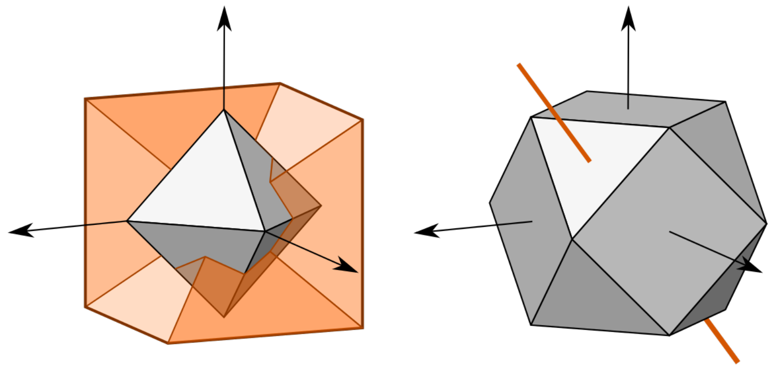

Figure 2.1: In this example $\mathcal{M}$ is uniform of rank 1 on 3 elements. On the left is the polytope $P_M^{\pm}$ together with $\Sigma_{\mathcal{M}}$. On the right is $P_{M^*}^{\pm}$ together with $\Sigma_{\mathcal{M}^*}$.

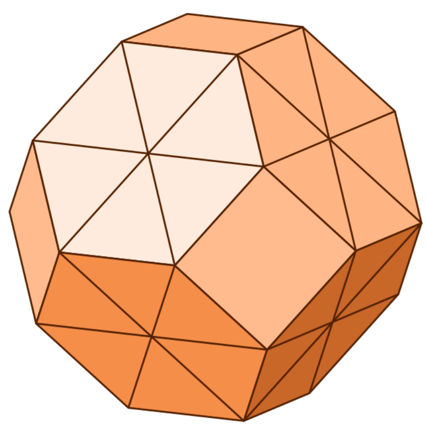

Figure 2.2: The intersection of the coarse subdivision of $\Sigma_{\mathcal{M}}$ with the boundary of $[-1,1]^6$. Here $\mathcal{M}$ is the graphic matroid of the complete graph K_4.

22]:

$$\mathsf{star}_{\Sigma_{\mathcal{M}^*}}(\mathbf{1}) := \{\sigma \in \Sigma_{\mathcal{M}^*} : \mathbf{1} \in \sigma\} = \mathcal{B}^+(\mathcal{M})$$

More generally, given any sign vector $s \in \{-1, 1\}^E$ corresponding to a tope of $\mathcal{M}$, we can recover the signed Bergman fans of Jürgens [31]:

$$\mathsf{star}_{\Sigma_{\mathcal{M}^*}}(s) = \mathcal{B}^s(\mathcal{M}).$$

The intersection $\Sigma_{\mathcal{M}} \cap [-1, 1]^E$ in $\mathbf{R}^E$ yields a polyhedral complex which da Silva and Moulton have called the *crinkled zonotope* of $\mathcal{M}^*$ [32]. The boundary of this polyhedral complex is a geometric realization of the order complex of the big face lattice of $\mathcal{M}$, which is known to be a sphere by the Topological Representation Theorem of Folkman and Lawrence [27, Theorem 5.2.1]. Within this sphere, the coordinate hyperplanes $x_e = 0$, $e \in E$ form a piecewise linear arrangement of pseudospheres that is represented by $\mathcal{M}^*$ [27, Definition 5.1.3].

This fan $\Sigma_{\mathcal{M}}$ can also be understood in the language of matroids over hyperfields due to Baker and Bowler [33]. Every oriented matroid $\mathcal{M}$ can be interpreted as a matroid $\mathcal{M}_{\mathbf{TR}}$ over the real tropical hyperfield $\mathbf{TR}$ defined by Viro in [34], with trivial valuation. The set of vectors of $\mathcal{M}_{\mathbf{TR}}$, in the sense of Anderson in [35], coincides exactly with the support of $\Sigma_{\mathcal{M}}$. This perspective suggests that it would be interesting to study the set of vectors of an oriented matroid $\mathcal{M}$ with nontrivial valuation.

Finally, we remark that $\Sigma_{\mathcal{M}}$ is defined in terms of vectors of $\mathcal{M}$ rather than covectors of $\mathcal{M}$, going against what seems to be the more common convention in the subject of oriented matroids. The reason is because it is $\Sigma_{\mathcal{M}}$, and not $\Sigma_{\mathcal{M}^*}$, that is a subfan of the outer normal fan of $P_M^{\pm}$.

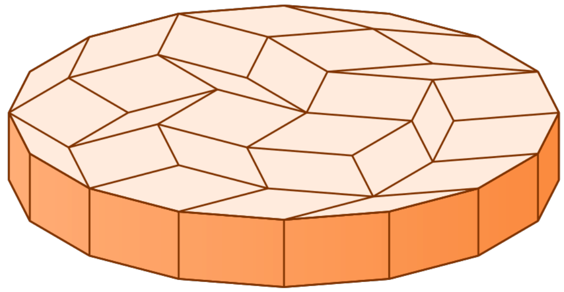

Figure 2.3: The intersection of the coarse subdivision of $\Sigma_{\mathcal{M}}$ with the boundary of $[-1,1]^9$. Here $\mathcal{M}$ is the oriented matroid dual to $\mathsf{Rin}(3,9)$. In general, when $\mathcal{M}$ is uniform, this intersection will always be a complex of parallelepipeds that is homeomorphic to a sphere.

The outer normal fan of $P_M^{\pm}$

Let M be *any* loop-free matroid of rank r on the ground set E, orientable or not. The goal of this section is to give a combinatorial interpretation to each cone in the outer normal fan of $P_M^{\pm}$, and to use this interpretation to describe the extreme rays of each cone in the fan:

Theorem 2.4.1. *There is a canonical bijection between the cones σ of the outer normal fan of $P_M^{\pm}$ and pairs $(X, \mathcal{D})$, where:*

- *X is a signed subset of E, and*
- *$\mathcal{D}$ is an initial sublattice of $\mathcal{L}_{M|S}$ where S is the support of X.*

Specifically, to such a pair $(X, \mathcal{D})$, we associate the cone $\sigma = \sigma(X, \mathcal{D})$ with extreme ray description given below:

$$\mathsf{cone}\left(\rho \in \mathbf{R}^E \ : \ \textit{for some } F \in \mathcal{D}^\dagger, \quad \begin{array}{ll} \rho_e = X_e & \textit{if } e \in F \\ \rho_e = \pm 1 & \textit{if } e \in \mathsf{cl}_M(F) \setminus F \\ \rho_e = 0 & \textit{if } e \in E \setminus \mathsf{cl}_M(F) \end{array}\right).$$

Here $\mathcal{D}^\dagger$ is the set of nonempty $F \in \mathcal{D}$ such that $\mathsf{cl}_M(F)$ is connected in M.

We remark that there is a fair bit of overlap with this section and Section 3.3 (d) of Fujishige's book [24], which considers similar questions regarding the structure of base polyhedra of submodular functions. Similar ideas also appear in the work of Kim in [25, Section 2]. Our context is different enough from these works that, except for some known standard results, we have decided to include full proofs of all the technical details.

Polytopes fixed by coordinate hyperplane reflections

Let $P \subseteq \mathbf{R}^E$ be a polytope that is fixed by all hyperplane reflections. Before we begin, we state some simple but important facts about P.

Definition 2.4.2. For a polytope $\tau \subseteq \mathbf{R}^E$, let $\mathsf{supp}(\tau)$ denote the set of all $f \in E$ such that τ is not contained in the hyperplane $x_f = 0$.

Proposition 2.4.3. *Let σ be a cone in the outer normal fan of P, let $\omega \in \mathsf{relint}(\sigma)$, and let $\tau = \tau_\omega$ denote the face of P that is maximized by ω.*

1. *For all $f \in \mathsf{supp}(\tau)$ we have*

$$\mathsf{sign}(\omega_f) = \begin{cases} +1, & \text{if } \tau \subseteq \{x : x_f \geq 0\} \\ -1, & \text{if } \tau \subseteq \{x : x_f \leq 0\} \\ 0, & \text{otherwise.} \end{cases}$$

2. *The vector $\omega' \in \mathbf{R}^E$ defined by*

$$\omega'_f = \begin{cases} \omega_f, & f \in \mathsf{supp}(\tau) \\ 0, & \text{otherwise} \end{cases}$$

   is a minimal-support element of $\mathsf{relint}(\sigma)$.

Proof. First we show (1). Let $f \in \mathsf{supp}(\tau)$, and let $x \in \tau$ so that $x_f \neq 0$. By the symmetry of P we also have $x - 2x_f\mathbf{e}_f \in P$, and so by definition of $\tau = \tau_\omega$ we have

$$\langle x - 2x_f\mathbf{e}_f, \omega\rangle \leq \langle x, \omega\rangle .$$

Therefore, we have $\omega_f x_f \geq 0$, with equality only if $x - 2x_f\mathbf{e}_f \in \tau$. Now, if $\tau \subseteq \{x : s_f x_f \geq 0\}$ for some $s_f \in \{\pm 1\}$, then we cannot have $x - 2x_f\mathbf{e}_f \in \tau$ which implies $\omega_f x_f > 0$ and hence $\mathsf{sign}(\omega_f) = \mathsf{sign}(x_f) = s_f$. Otherwise, we may find $y \in \tau \cap \{x : x_f > 0\}$ and $z \in \tau \cap \{x : x_f < 0\}$. Hence $\omega_f y_f \geq 0$ and $\omega_f z_f \geq 0$, which is only possible if $\omega_f = 0$.

Next, let ω' be as in (2). To see that ω' lies in $\mathsf{relint}(\sigma)$, we show that if $x \in \tau$ then $\langle \omega', x\rangle = \langle \omega, x\rangle$ and if $x' \in P \setminus \tau$, then $\langle \omega', x'\rangle < \langle \omega', x\rangle$ for any $x \in \tau$. The first assertion follows from the definition of $\mathsf{supp}(\tau)$ and the definition of ω'. For the second assertion, let $x \in \tau$ and let $x' \in P \setminus \tau$. Let $y' \in P$ be the point obtained from x' by negating the components outside of $\mathsf{supp}(\tau)$; that is, $y' = x' - 2\sum_{f \in E \setminus \mathsf{supp}(\tau)} x'_f \mathbf{e}_f$. We have

$$\begin{aligned}
2\langle \omega', x'\rangle &= \langle \omega', x'\rangle + \langle \omega', y'\rangle \\
&= \langle \omega, x'\rangle + \langle \omega' - \omega, x'\rangle + \langle \omega, y'\rangle + \langle \omega' - \omega, y'\rangle \\
&< \langle \omega, x\rangle + \langle \omega' - \omega, x'\rangle + \langle \omega, x\rangle + \langle \omega' - \omega, y'\rangle \\
&= 2\langle \omega, x\rangle + \langle \omega' - \omega, x' + y'\rangle \\
&= 2\langle \omega, x\rangle \\
&= 2\langle \omega', x\rangle
\end{aligned}$$

where the second-to-last equality holds since $\mathsf{supp}(\omega' - \omega) \subseteq E \setminus \mathsf{supp}(\tau)$ while $\mathsf{supp}(x' + y') \subseteq \mathsf{supp}(\tau)$. □

Let G be the subgroup of $\mathsf{GL}(\mathbf{R}^E)$ generated be coordinate hyperplane reflections;

that is, diagonal matrices with ± 1 entries along the diagonal. Observe that the group G acts on the set of outer normal cones of P: for each $g \in G$ and cone σ, we have that

$$g\sigma := \{g\omega : \omega \in \sigma\}$$

is also a cone in the outer normal fan of P.

Proposition 2.4.4. *Let $\omega \in \mathsf{relint}(\sigma)$ be a minimal-support element of a cone σ in the outer normal fan of P.*

1. *The stabilizer of σ under this action, given by*

$$\mathsf{stab}_G(\sigma) := \{g \in G : g\sigma = \sigma\},$$

   is generated by reflections about $x_f = 0$ where $f \in E \setminus \mathsf{supp}(\omega)$.

2. *Let $G \cdot \sigma := \{g\sigma : g \in G\}$ denote the orbit of σ under G. The function*

$$\begin{aligned} G \cdot \sigma &\longrightarrow \{-1, 1\}^{\mathsf{supp}(\omega)} \\ g\sigma &\longmapsto \mathsf{sign}(g\omega) \end{aligned}$$

   is a bijection, where we define $(\mathsf{sign}(g\omega))_f$ to be the sign of $(g\omega)_f$.

Proof. (1) Suppose $f \in E \setminus \mathsf{supp}(\omega)$, and suppose $g \in G$ is the reflection about $x_f = 0$. Then $g\omega = \omega$, which implies $g\sigma = \sigma$ since ω lies in the relative interior of exactly one normal cone of P. This shows that every such reflection is an element of $\mathsf{stab}_G(\sigma)$.

Conversely, suppose $g \in G$ satisfies $g\sigma = \sigma$. Then $g\omega$ and ω are both elements of the relative interior of σ. In particular, the midpoint $(\omega + g\omega)/2$ lies in the relative interior of σ. Since both $g\omega$ and ω are in fact *minimal-support* elements of $\mathsf{relint}(\sigma)$, we must further have that ω and $g\omega$ agree in sign since otherwise $(\omega + g\omega)/2$ would have even smaller support. We therefore conclude that in any minimal representation $g = g_1 g_2 \cdots g_k$ of g by a product of coordinate hyperplane reflections, each g_i must be a reflection about $x_f = 0$ for some $f \in E \smallsetminus \mathsf{supp}(\omega)$.

(2) It is immediate that this function is surjective. Injectivity follows from

$$|G \cdot \sigma| = |G| \,/\, |\mathsf{stab}_G(\sigma)| = 2^{|E|}/2^{|E \smallsetminus \mathsf{supp}(\omega)|} = 2^{|\mathsf{supp}(\omega)|}.$$

□

The aspect of Proposition 2.4.4 that is used later in the paper is the following corollary, which allows for some simplification of notation:

Proposition 2.4.5. *The bijection of Proposition 2.4.4 (2) determines a bijection between cones in the outer normal fan of P, and pairs (X, σ) where:*

- *The item $X \in \{-1, 0, 1\}^E$ is a signed subset of E, and*
- *The item σ is a cone in the outer normal fan of P with the property that there exists a minimal-support element $\omega \in \mathsf{relint}(\sigma) \cap \mathbf{R}^E_{\geq 0}$ satisfying $\mathsf{supp}(\omega) = \mathsf{supp}(X)$.*

Initial matroids

The next step is to attach a matroid to each face of $P^{\pm}_M$. These matroids include not only the *initial matroids* of M as defined in Ardila-Klivans [21], but also the initial matroids of $M|S$ where S ranges over all subsets of E.

Definition 2.4.6. Let $\omega \in \mathbf{R}^E$ and let $S = \mathsf{supp}(\omega)$. We define the *flag of ω* to be the unique chain of strictly increasing subsets of S

$$\emptyset = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_k = S$$

such that $|\omega_e|$ is constant over all $e \in E_i \smallsetminus E_{i-1}$ for $i = 1, 2, \ldots, k$, and

$$|\omega_{e_1}| > |\omega_{e_2}| > \cdots > |\omega_{e_k}| > 0$$

whenever $e_1 \in E_1 \smallsetminus E_0,\ e_2 \in E_2 \smallsetminus E_1,\ \ldots,\ e_k \in E_k \smallsetminus E_{k-1}$.

We define the matroid M^ω on S as follows:

$$M^\omega := \bigoplus_{i=1}^{k} (M \mid E_i)/E_{i-1}. \tag{2.2}$$

where $\emptyset = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_k = S$ is the flag of ω as in Defintion 2.4.6.

If $\omega \in \mathbf{R}^E$ is viewed as a linear objective function, we denote by τ_ω the face of $P_M^\pm$ that is maximized by ω.

Proposition 2.4.7. *Let $B \subseteq S$. The following are equivalent:*

1. *B is a basis of M^ω.*
2. *B is a basis of $M|S$ and*

$$\sum_{f \in B} |\omega_f| \geq \sum_{f \in B'} |\omega_f|$$

for all other bases B' of $M|S$.

3. *There exists a vertex v of τ_ω such that $B = \mathsf{supp}(v) \cap S$.*

Proof. The equivalence of (1) and (2) is well-known, see [21, Proposition 2].

(2) implies (3): Let B be a basis of $M|S$ as in (2), and let v be a vertex of $P_M^\pm$ such that $\mathsf{sign}(v_f) = \mathsf{sign}(\omega_f)$ for all $f \in S$, and $B \subseteq \mathsf{supp}(v)$. Such a v exists because any basis of $M|S$ can be extended to a basis of M. We have $B \subseteq S$, and equality must hold in the containment $B \subseteq \mathsf{supp}(v) \cap S$ since otherwise B would be too small to be a basis of $M|S$. It therefore remains to show that v is a vertex of τ_ω. Choose any other vertex w of $P_M^\pm$. Then $\mathsf{supp}(w) \cap S$ is an independent set in $M|S$, which

can be extended to a basis B' of $M|S$, so that by (2) we have

$$\langle v, \omega \rangle = \sum_{f \in B} |\omega_f| \geq \sum_{f \in B'} |\omega_f| \geq \sum_{f \in \mathsf{supp}(w) \cap S} |\omega_f| \geq \langle w, \omega \rangle .$$

(3) implies (2): Write $B = \mathsf{supp}(v) \cap S$ for some vertex v of τ_ω. Let B' be a basis of $M|S$. Then we can find a vertex w of $P_M^{\pm}$ such that $B' = \mathsf{supp}(w) \cap S$ and $\mathsf{sign}(w_f) = \mathsf{sign}(\omega_f)$ for all $f \in B'$. Hence

$$\sum_{f \in B'} |\omega_f| = \langle w, \omega \rangle \leq \langle v, \omega \rangle = \sum_{f \in B} |\omega_f| .$$

□

Proposition 2.4.8. *The non-loops of M^ω are precisely those $f \in E$ such that the image of the projection*

$$\pi_f : \{\textit{vertices of } \tau_\omega\} \to \{-1, 0, 1\}$$
$$v \mapsto v_f$$

contains 1 or -1 but not both. Furthermore, for each such non-loop f, the unique nonzero element in the image of π_f is equal to $\mathsf{sign}(\omega_f)$.

Proof. First, suppose that $f \in E$ is such that the image of π_f contains 1 or -1 but not both. Denote this sign by s_f. This means $\tau_\omega \subseteq \{x : s_f x_f \geq 0\}$ and there exists a vertex v of τ_ω for which $\mathsf{sign}(v_f) = s_f$. By Proposition 2.4.3 we must also have $\mathsf{sign}(\omega_f) = s_f \neq 0$ and therefore f lies in the ground set of M^ω. By Proposition 2.4.7, $\mathsf{supp}(v) \cap S$ is a basis of M^ω containing f, and hence f is not a loop of M^ω.

Conversely, assume that f is a non-loop of M^ω. Proposition 2.4.7 then implies that $f \in \mathsf{supp}(\tau_\omega)$. Since $\omega_f \neq 0$, we must have by Proposition 2.4.3 that $\tau_\omega \subseteq \{x : \mathsf{sign}(\omega_f) x_f \geq 0\}$. In other words, the image of π_f contains $\mathsf{sign}(\omega_f)$ but not $-\mathsf{sign}(\omega_f)$. □

Proposition 2.4.9. *For any two $\omega, \omega' \in \mathbf{R}^E$, we have $\tau_\omega = \tau_{\omega'}$ if and only if the loop-free part of M^ω equals the loop-free part of $M^{\omega'}$, and $\mathsf{sign}(\omega_f) = \mathsf{sign}(\omega'_f)$ for all non-loops f of M^ω.*

Proof. The forward implication is equivalent to the assertion that for any $\omega \in \mathbf{R}^E$, the data

$$\left(M^\omega \smallsetminus (\text{loops of } M^\omega), (\mathsf{sign}(\omega_f))_{f \in \{\text{non-loops of } M^\omega\}}\right)$$

depends only on τ_ω. The equivalence of (1) and (3) in Proposition 2.4.7 implies that the set of bases of M^ω equals

$$\{\mathsf{supp}(v) \cap \{\text{non-loops of } M^\omega\} : v \text{ is a vertex of } \tau_\omega\}.$$

Proposition 2.4.8 shows how to recover the set of non-loops of M^ω from τ_ω, as well as $\mathsf{sign}(\omega_f)$ for each non-loop f of M^ω.

Now suppose the loop-free part of M^ω equals the loop-free part of $M^{\omega'}$, and $\mathsf{sign}(\omega_f) = \mathsf{sign}(\omega'_f)$ whenever f is a non-loop of M^ω. By Propositions 2.4.7 and 2.4.8, we recover the face τ_ω by taking the convex hull of all lattice points $v \in \{-1, 0, 1\}^E$ such that $\mathsf{supp}(v) \cap S$ is a basis of M^ω which extends to a basis $\mathsf{supp}(v)$ of M, and $\mathsf{sign}(v_f) = \mathsf{sign}(\omega_f)$ whenever both are nonzero (in which case f is a non-loop of M^ω). Since this procedure is the same for both ω and ω', we conclude $\tau_\omega = \tau_{\omega'}$. $\square$

We finish off this section by showing that if $\omega \in \mathsf{relint}(\sigma)$ has minimal support, then M^ω is loop-free.

Proposition 2.4.10. *Let σ be a cone in the outer normal fan of $P_M^\pm$, and let $\omega \in \mathsf{relint}(\sigma)$ have minimal support. Then M^ω is loop-free.*

Proof. Suppose $\omega \in \mathsf{relint}(\sigma)$ has minimal support. If $f \in \mathsf{supp}(\omega)$, then we must have $\tau_\omega \subseteq \{x : \mathsf{sign}(\omega_f)x_f \geq 0\}$ by Proposition 2.4.3 (1), and, furthermore, we must also have $f \in \mathsf{supp}(\tau_\omega)$ because if not then we could find an even-smaller-support element of $\mathsf{relint}(\sigma)$ by Proposition 2.4.3 (2). This implies, by Proposition 2.4.8, that f is a non-loop of M^ω. $\square$

Details of the bijection

In light of Proposition 2.4.5, in order to establish the bijection of Theorem 2.4.1 it suffices to establish, for each $S \subseteq E$, the following restricted bijection between:

- Initial sublattices $\mathcal{D}$ of $\mathcal{L}_{M|S}$, and
- Outer normal cones σ of $P_M^{\pm}$ such that there exists $\omega \in \mathsf{relint}(\sigma) \cap \mathbf{R}_{\geq 0}^E$ with the property that ω is a minimal-support element of $\mathsf{relint}(\sigma)$ and $S = \mathsf{supp}(\omega)$.

The map $\mathcal{D} \mapsto \sigma_{\mathcal{D}}$ Let $\emptyset \subsetneq E_1 \subsetneq \cdots \subsetneq E_k$ be a maximal chain in $\mathcal{D}$, and choose any $\omega \in \mathsf{relint}(\mathsf{cone}(\mathbf{e}_{E_1}, \mathbf{e}_{E_2}, \ldots, \mathbf{e}_{E_k}))$. Then let $\sigma_{\mathcal{D}}$ be the unique cone in the outer normal fan of $P_M^{\pm}$ such that $\omega \in \mathsf{relint}(\sigma_{\mathcal{D}})$.

The map $\sigma \mapsto \mathcal{D}_\sigma$ Choose $\omega \in \mathsf{relint}(\sigma) \cap \mathbf{R}_{\geq 0}^E$, which, by Proposition 2.4.3 (2), we may assume to be a minimal-support element of $\mathsf{relint}(\sigma)$. Define $\mathcal{D}_\sigma$ to be the collection of all unions $F = S_1 \cup S_2 \cup \cdots \cup S_m$ such that each S_i is the ground set of a connected component of M^ω, and $\mathsf{rk}_{M^\omega}(F) = \mathsf{rk}_M(F)$.

Lemma 2.4.11. *These maps are well-defined and are inverses of eachother.*

The proof of this lemma is carried out over the next four subsections.

Sublattices of a boolean lattice, posets, and linear extensions

Let $\mathcal{D}$ be a sublattice of the boolean lattice 2^S, where S is some finite set. Here we collect some standard facts (without proofs) about posets and sublattices of a boolean lattice. We refer the reader to [36] for a general introduction to the topic.

Proposition 2.4.12. *[24, Corollary 3.10] The partition*

$$\Pi_{\mathscr{C}} = \{E_i \smallsetminus E_{i-1} : i = 1, 2, \ldots, k\}$$

of S is the same for every maximal chain $\mathscr{C} : \emptyset = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_k = S$ of $\mathcal{D}$. Furthermore, every element of $\mathcal{D}$ can be written as a union of parts from $\Pi_{\mathscr{C}}$.

We may therefore write $\Pi_{\mathscr{C}}$ appearing in the above proposition as $\Pi_{\mathcal{D}}$.

Definition 2.4.13. We define the poset $\mathbf{P}_{\mathcal{D}}$ as follows: the ground set of this poset is $\Pi_{\mathcal{D}}$, and two parts $S_i, S_j \in \Pi_{\mathcal{D}}$ satisfy $S_i \leq S_j$ in $\mathbf{P}_{\mathcal{D}}$ if and only if every element of $\mathcal{D}$ containing S_i also contains S_j.

Note that antisymmetry holds for $\mathbf{P}_{\mathcal{D}}$ because if $1 \leq i < j \leq k$ and $S_i = E_i \smallsetminus E_{i-1} \in \Pi_{\mathcal{D}}$ and $S_j = E_j \smallsetminus E_{j-1} \in \Pi_{\mathcal{D}}$, then E_{j-1} is an element of $\mathcal{D}$ containing S_i but not S_j.

Definition 2.4.14. A *linear extension* $\mathscr{L}$ of $\mathbf{P}_{\mathcal{D}}$ is a total ordering

$$\mathscr{L} : S_{i_1} < S_{i_2} < \cdots < S_{i_k}$$

of the elements of $\Pi_{\mathcal{D}}$ such that $S_i < S_j$ in $\mathbf{P}_{\mathcal{D}}$ implies $S_i < S_j$ in $\mathscr{L}$.

Proposition 2.4.15. *If $\mathscr{L} : S_{i_1} < S_{i_2} < \cdots < S_{i_k}$ is a linear extension of $\mathbf{P}_{\mathcal{D}}$, then*

$$\emptyset \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_k = S$$

is a maximal chain of $\mathcal{D}$, where $E_j := S_{i_1} \cup \cdots \cup S_{i_j}$ for $j = 1, 2, \ldots, k$. Furthermore, every maximal chain of $\mathcal{D}$ arises this way.

Definition 2.4.16. We say that two linear extensions $\mathscr{L}, \mathscr{L}'$ of $\mathbf{P}_{\mathcal{D}}$ are *adjacent* if there exists a unique pair $S_i, S_j \in \Pi_{\mathcal{D}}$ such that $S_i < S_j$ in $\mathscr{L}$ and $S_i > S_j$ in $\mathscr{L}'$.

Proposition 2.4.17. *Let $\mathscr{L}, \mathscr{L}'$ be two linear extensions of $\mathbf{P}_{\mathcal{D}}$. Then there exists a sequence of linear extensions $\mathscr{L} = \mathscr{L}_0, \mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_t = \mathscr{L}'$ such that $\mathscr{L}_{i-1}$ and $\mathscr{L}_i$ are adjacent for all $i = 1, 2, \ldots, t$.*

The map $\mathcal{D} \mapsto \sigma_{\mathcal{D}}$ is well-defined.

We remark that much of the work that appears in this section is based on [25, Section 2].

Let $\mathcal{D}$ be an initial sublattice of $\mathcal{L}_{M|S}$ for some $S \subseteq E$. Given a linear extension $\mathscr{L} : S_{i_1} < S_{i_2} < \cdots < S_{i_k}$ of $\mathbf{P}_{\mathcal{D}}$, define the cone

$$\mathsf{cone}(\mathscr{L}) := \mathsf{cone}(\mathbf{e}_{E_1}, \mathbf{e}_{E_2}, \ldots, \mathbf{e}_{E_k})$$

where $E_j := S_{i_1} \cup \cdots \cup S_{i_j}$ for $j = 1, 2, \ldots, k$.

Proposition 2.4.18. *Given such a linear extension $\mathscr{L}$, the matroid M^ω is the same for all $\omega \in \mathsf{relint}(\mathsf{cone}(\mathscr{L}))$.*

Proof. This holds since the flag of ω is $\emptyset \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_k = S$, and it is the flag of ω which determines M^ω. $\square$

Proposition 2.4.19. *Let $\mathscr{L}$ be a linear extension of $\mathbf{P}_{\mathcal{D}}$ and let $\omega \in \mathsf{relint}(\mathsf{cone}(\mathscr{L}))$. Then M^ω is loop-free.*

Proof. Let $\emptyset = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_k = S$ be the flag of ω. Then by Proposition 2.4.15, each $E_i \in \mathcal{D}$ and is therefore is a flat of $M|S$. In the decomposition

$$M^\omega = \bigoplus_{i=1}^{k} (M|E_i)/E_{i-1},$$

each component is loop-free since M is loop-free and the E_i's are flats. Therefore, M^ω is loop-free. $\square$

Proposition 2.4.20. *Let $S \subsetneq E$. Two flats F_1, F_2 of $M|S$ form a modular pair in $M|S$ if and only if the matroid $N := M|(F_1 \cup F_2)/(F_1 \cap F_2)$ admits a decomposition*

$$N = N_1 \oplus N_2$$

where $N_1 = N|(F_1 \smallsetminus (F_1 \cap F_2))$ and $N_2 = N|(F_2 \smallsetminus (F_1 \cap F_2))$.

Proof. The equality $N = N_1 \oplus N_2$ holds if and only if

$$\mathsf{rk}_N((F_1 \cup F_2) \smallsetminus (F_1 \cap F_2)) = \mathsf{rk}_N(F_1 \smallsetminus (F_1 \cap F_2)) + \mathsf{rk}_N(F_2 \smallsetminus (F_1 \cap F_2)),$$

which is equivalent to

$$\mathsf{rk}_M(F_1 \cup F_2) - \mathsf{rk}_M(F_1 \cap F_2) = \mathsf{rk}_M(F_1) - \mathsf{rk}_M(F_1 \cap F_2) + \mathsf{rk}_M(F_2) - \mathsf{rk}_M(F_1 \cap F_2)$$

or just

$$\mathsf{rk}_M(F_1 \cup F_2) + \mathsf{rk}_M(F_1 \cap F_2) = \mathsf{rk}_M(F_1) + \mathsf{rk}_M(F_2).$$

□

Proposition 2.4.21. *Let $\mathscr{L}, \mathscr{L}'$ be two linear extensions of $\mathbf{P}_{\mathcal{D}}$ such that $\mathscr{L}$ and $\mathscr{L}'$ are adjacent. Let $\omega \in \mathsf{relint}(\mathsf{cone}(\mathscr{L}))$ and let $\omega' \in \mathsf{relint}(\mathsf{cone}(\mathscr{L}'))$. Then $M^{\omega} = M^{\omega'}$.*

Proof. Let

$$\emptyset = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{i-1} \subsetneq E_i \subsetneq E_{i+1} \subsetneq \cdots \subsetneq E_k = S$$
$$\emptyset = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{i-1} \subsetneq E_i' \subsetneq E_{i+1} \subsetneq \cdots \subsetneq E_k = S$$

be the flags of ω and ω', respectively, so that $E_i \neq E_i'$. These are maximal chains of

$\mathcal{D}$ by Proposition 2.4.15. In particular, we must have

$$E_i \cap E_i' = E_{i-1}$$
$$E_i \cup E_i' = E_{i+1}$$

since otherwise we could make either of these chains even longer. Now, the procedures for obtaining bases of M^{ω} and $M^{\omega'}$ are as follows. In both cases, a basis of $M|E_1$ is extended to a basis of $M|E_2$, then to a basis of $M|E_3$, and so on, until a basis of $M|E_{i-1}$ is reached. This basis is then extended to $M|E_i$ in the case of M^{ω}, or $M|E_i'$ in the case of $M^{\omega'}$. From there, for both M^{ω} and $M^{\omega'}$, the result is then extended to a basis of $M|E_{i+1}$, then to a basis of $M|E_{i+2}$, and so on, until a basis of $M|E_k = M|S$ is reached. These two procedures yield the same bases if and only if the matroid $N := M|E_{i+1}/E_{i-1}$ admits a decomposition

$$N = N_1 \oplus N_2,$$

where $N_1 = N|(E_i \setminus E_{i-1})$ and $N_2 = N|(E_i' \setminus E_{i-1})$ This happens if and only if E_i, E_i' form a modular pair of flats in $M|S$ by Proposition 2.4.20. This is indeed the case, since both $E_i, E_i' \in \mathcal{D}$ by assumption. □

Corollary 2.4.22. *[25, Proposition 2.5] The map $\mathcal{D} \mapsto \sigma_{\mathcal{D}}$ is well defined.*

Proof. Suppose $\omega, \omega' \in \mathbf{R}^E_{\geq 0}$ are two vectors such that the flags of both are maximal chains of $\mathcal{D}$. Then by Proposition 2.4.15, there exists two linear extensions $\mathscr{L}, \mathscr{L}'$ of $\mathbf{P}_{\mathcal{D}}$ such that $\omega \in \mathsf{relint}(\mathsf{cone}(\mathscr{L}))$ and $\omega' \in \mathsf{relint}(\mathsf{cone}(\mathscr{L}'))$. If $\mathscr{L} = \mathscr{L}'$ then by Proposition 2.4.18 we have $M^{\omega} = M^{\omega'}$. Otherwise, by Proposition 2.4.17, $\mathscr{L}$ and $\mathscr{L}'$ are connected by a sequence of linear extensions $\mathscr{L} = \mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_t = \mathscr{L}'$ of $\mathbf{P}_{\mathcal{D}}$ such that every consecutive pair of linear extensions in this sequence are adjacent. Applying Proposition 2.4.21 to $\omega_i := \omega(\mathscr{L}_i)$ for $i = 0, 1, 2, \ldots, t$, we see that $M^{\omega} =$

$M^{\omega_0} = M^{\omega_1} = \cdots = M^{\omega_t} = M^{\omega'}$. Moreover, by Proposition 2.4.15, M^ω and $M^{\omega'}$ are loop-free. By Proposition 2.4.9, we therefore conclude $\tau_\omega = \tau_{\omega'}$. This is equivalent to the assertion that both ω, ω' lie in the relative interior of the same cone σ in the outer normal fan of $P_M^{\pm}$. $\square$

The map $\sigma \mapsto \mathcal{D}_\sigma$ is well defined

Let σ be a cone in the outer normal fan of $P_M^{\pm}$ such that there exists some minimal-support $\omega \in \mathsf{relint}(\sigma) \cap \mathbf{R}^E_{\geq 0}$. Then by Proposition 2.4.10 M^ω is loop-free, so that by Theorem 2.4.9, M^ω does not depend on ω but only σ. Let S denote the ground set of M^ω, and let Π denote the partition of S into ground sets of the connected components of M^ω.

We define $\mathcal{D} = \mathcal{D}_\sigma$ to be the collection of unions $F = S_1 \cup S_2 \cup \cdots \cup S_m$ of parts of Π such that $\mathsf{rk}_{M^\omega}(F) = \mathsf{rk}_M(F)$. It is always true $\mathsf{rk}_{M^\omega}(F) \leq \mathsf{rk}_M(F)$ for any $F \subseteq S$, so $\mathcal{D}$ captures those F for which equality holds.

Proposition 2.4.23. *The set system $\mathcal{D}$ forms an initial sublattice of $M|S$.*

Proof. To show $\mathcal{D}$ is an initial sublattice of $M|S$, we need to show four things.

(1) Suppose $F_1, F_2 \subseteq S$ satisfy $F_1 \cup F_2 \in \mathcal{D}$, $F_1 \cap F_2 \in \mathcal{D}$, and

$$\mathsf{rk}_M(F_1) + \mathsf{rk}_M(F_2) = \mathsf{rk}_M(F_1 \cup F_2) + \mathsf{rk}_M(F_1 \cap F_2).$$

We want to show $F_1 \in \mathcal{D}$ and $F_2 \in \mathcal{D}$ and both F_1, F_2 are unions of parts of Π. To see that $F_1, F_2 \in \mathcal{D}$, observe that

$$\begin{aligned}
\mathsf{rk}_{M^\omega}(F_1) + \mathsf{rk}_{M^\omega}(F_2) &= \mathsf{rk}_{M^\omega}(F_1 \cup F_2) + \mathsf{rk}_{M^\omega}(F_1 \cap F_2) \\
&= \mathsf{rk}_M(F_1 \cup F_2) + \mathsf{rk}_M(F_1 \cap F_2) \\
&= \mathsf{rk}_M(F_1) + \mathsf{rk}_M(F_2) \\
&\geq \mathsf{rk}_{M^\omega}(F_1) + \mathsf{rk}_{M^\omega}(F_2)
\end{aligned}$$

and so equality holds throughout. In particular, we have

$$\begin{aligned}\mathsf{rk}_{M^\omega}(F_1) &= \mathsf{rk}_M(F_1)\\ \mathsf{rk}_{M^\omega}(F_2) &= \mathsf{rk}_M(F_2),\end{aligned}$$

that is, F_1 and F_2 both lie in $\mathcal{D}$.

We next show that F_1 and F_2 must be unions of parts of Π. Since both $F_1 \cup F_2$ and $F_1 \cap F_2$ lie in $\mathcal{D}$, the only way that this cannot happen is if there is some connected component M' of M^ω on the ground set $S' \in \Pi$ which contains some $e \in S' \cap (F_1 \setminus F_2)$ and $f \in S' \cap (F_2 \setminus F_1)$. Now since M' is connected, there is a circuit C of M' containing both e and f. Choose a basis B of $M^\omega|(F_1 \cup F_2)$ such that $f \in B$ and C is the fundamental circuit of $B \cup e$. Note that $B \cap F_1 \cap F_2$ is a basis of $M^\omega|(F_1 \cap F_2)$, because $F_1 \cap F_2$ is a union of parts in Π. We therefore conclude

$$\begin{aligned}\mathsf{rk}_M(F_1 \cup F_2) + \mathsf{rk}_M(F_1 \cap F_2) &= \mathsf{rk}_{M^\omega}(F_1 \cup F_2) + \mathsf{rk}_{M^\omega}(F_1 \cap F_2)\\ &= |B \cap F_1 \cap F_2| + |B \cap (F_1 \cup F_2)|\\ &= |B \cap F_1| + |B \cap F_2|\\ &< |(B \cup e) \cap F_1| + |B \cap F_2|\\ &\leq \mathsf{rk}_{M^\omega}(F_1) + \mathsf{rk}_{M^\omega}(F_2)\\ &= \mathsf{rk}_M(F_1) + \mathsf{rk}_M(F_2)\\ &= \mathsf{rk}_M(F_1 \cup F_2) + \mathsf{rk}_M(F_1 \cap F_2)\end{aligned}$$

which is a contradiction.

(2) Let $F_1, F_2 \in \mathcal{D}$. We want to show $F_1 \cap F_2$ and $F_1 \cup F_2$ are also in $\mathcal{D}$, and

$$\mathsf{rk}_M(F_1) + \mathsf{rk}_M(F_2) = \mathsf{rk}_M(F_1 \cup F_2) + \mathsf{rk}_M(F_1 \cap F_2).$$

We have

$$\begin{aligned}
\mathsf{rk}_{M^\omega}(F_1 \cup F_2) + \mathsf{rk}_{M^\omega}(F_1 \cap F_2) &\leq \mathsf{rk}_M(F_1 \cup F_2) + \mathsf{rk}_M(F_1 \cap F_2) \\
&\leq \mathsf{rk}_M(F_1) + \mathsf{rk}_M(F_2) \\
&= \mathsf{rk}_{M^\omega}(F_1) + \mathsf{rk}_{M^\omega}(F_2) \\
&= \mathsf{rk}_{M^\omega}(F_1 \cup F_2) + \mathsf{rk}_{M^\omega}(F_1 \cap F_2)
\end{aligned}$$

where the second inequality holds by submodularity of the rank function. So equality holds throughout, which establishes all three assertions of (2).

(3) We want to show every $F \in \mathcal{D}$ is a flat of $M|S$. This is an immediate consequence of the fact that F is a union of parts of Π, which means, since M^ω is loop-free, that every $e \in S \smallsetminus F$ satisfies

$$\mathsf{rk}_M(F) + 1 \geq \mathsf{rk}_M(F \cup e) \geq \mathsf{rk}_{M^\omega}(F \cup e) = \mathsf{rk}_{M^\omega}(F) + 1 = \mathsf{rk}_M(F) + 1.$$

So F is indeed a flat of $M|S$.

(4) Finally, we want to show $\emptyset \in \mathcal{D}$ and $S \in \mathcal{D}$. That $\emptyset \in \mathcal{D}$ is clear from the definition, as $\mathsf{rk}_{M^\omega}(\emptyset) = \mathsf{rk}_M(\emptyset) = 0$ and $\emptyset$ is the empty union of the connected components of M^ω. To see that $S \in \mathcal{D}$, since S is the ground set of M^ω it is obviously a union of the connected components of M^ω. The fact that $\mathsf{rk}_{M^\omega}(S) = \mathsf{rk}_M(S)$ follows immediately from Proposition 2.4.7. $\square$

Injectivity and surjectivity

We are going to show that our map is a bijection by showing that the maps $\sigma \mapsto \mathcal{D}_\sigma$ and $\mathcal{D} \mapsto \sigma_{\mathcal{D}}$ are inverses of eachother.

Proposition 2.4.24. *Let $\omega \in \mathbf{R}^E$ and let $\emptyset \subsetneq E_1 \subsetneq \cdots \subsetneq E_k$ denote the flag of ω. Then $\mathsf{rk}_{M^\omega}(E_j) = \mathsf{rk}_M(E_j)$ for all $j = 1, 2, \ldots, k$.*

Proof. By Proposition 2.4.7 applied to the vector $\omega^{(j)}$ where

$$\omega_e^{(j)} = \begin{cases} \omega_e, & e \in E_j \\ 0, & e \notin E_j, \end{cases}$$

we see that, for $j = 1, 2, \ldots, k$, we have $\mathsf{rk}_{M^\omega}(E_j) = \mathsf{rk}_{M^{\omega^{(j)}}}(E_j) = \mathsf{rk}_M(E_j)$. $\square$

Proposition 2.4.25. *We have* $\sigma = \sigma_{\mathcal{D}_\sigma}$.

Proof. Let σ be a cone in the outer normal fan of $P_M^{\pm}$ whose relative interior intersects $\mathbf{R}_{\geq 0}^E$. Among all $\omega \in \mathsf{relint}(\sigma) \cap \mathbf{R}_{\geq 0}^E$ such that ω is a minimal-support element of $\mathsf{relint}(\sigma)$, choose ω so that the flag $\emptyset = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_k = S$ of ω is as long as possible. We start by showing that with this carefully chosen ω, the decomposition

$$M^\omega = \bigoplus_{i=1}^{k} (M|E_i)/E_{i-1}$$

is a decomposition of M^ω into connected components. Suppose this was not the case, so that there exists some i such that $(M|E_i)/E_{i-1} = M_1' \oplus M_2'$. Let E_1', E_2' denote the ground sets of M_1', M_2', respectively. For sufficiently small $\varepsilon > 0$, the vector $\omega' \in \mathsf{relint}(\sigma)$ defined by

$$\omega_e' = \begin{cases} \omega_e, & e \notin E' \\ (1+\varepsilon)\omega_e & e \in E' \end{cases}$$

optimizes the same face of $P_M^{\pm}$ as ω but has a strictly longer flag than ω. This contradicts maximality of the flag of ω. We therefore conclude that

$$\Pi := \{E_i \smallsetminus E_{i-1} : i = 1, 2, \ldots, k\}$$

is the partition of S into the ground sets of the connected components of M^ω.

If we can show that $\omega \in \mathsf{relint}(\sigma_{\mathcal{D}_\sigma})$, then we are done. By definition, $\sigma_{\mathcal{D}_\sigma}$ is defined to be the unique cone in the outer normal fan of $P_M^{\pm}$ containing the relative interior of $\mathsf{cone}(\mathbf{e}_{E'_1}, \dots, \mathbf{e}_{E'_k})$ where $\emptyset \subsetneq E'_1 \subsetneq \cdots \subsetneq E'_k$ is any maximal chain of $\mathcal{D}_\sigma$. Hence, it suffices to show that $\emptyset \subsetneq E_1 \subsetneq \cdots \subsetneq E_k$ is a maximal chain of $\mathcal{D}_\sigma$.

Now, $\mathcal{D}_\sigma$ is defined to be the collection of unions F of parts of Π such that $\mathsf{rk}_{M^\omega}(F) = \mathsf{rk}_M(F)$. By Proposition 2.4.24, each $E_j \in \mathcal{D}_\sigma$. Moreover, the length k of the chain $\emptyset \subsetneq E_1 \subsetneq \cdots \subsetneq E_k$ is equal to the cardinality of Π, which implies that this chain must be a maximal chain of $\mathcal{D}_\sigma$. $\square$

Proposition 2.4.26. *Let $\mathcal{D}', \mathcal{D}''$ be two initial sublattices of $\mathcal{L}_{M|S}$, such that $\emptyset = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_k = S \subseteq E$ is a maximal chain of both. Then $\mathcal{D}' = \mathcal{D}''$.*

Proof. The statement is symmetric in $\mathcal{D}'$ and $\mathcal{D}''$, so it suffices to show $\mathcal{D}' \subseteq \mathcal{D}''$. Let $F \in \mathcal{D}'$. By Proposition 2.4.12, we may write $F = S_{i_1} \cup \cdots \cup S_{i_m}$ where each $S_{i_j} = E_{i_j} \smallsetminus E_{i_j - 1}$. We may further assume that $E_{i_1} \subsetneq E_{i_2} \subsetneq \cdots \subsetneq E_{i_m}$, so that

$$F_j := S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_j} = F \cap E_{i_j} \text{ for all } j = 1, 2, \dots, m.$$

Note that these F_j's lie in $\mathcal{D}'$.

We show by induction on $j = 0, 1, \dots, m$ that F_j lies in $\mathcal{D}''$. The case $j = 0$ holds since $\emptyset \in \mathcal{D}''$. Therefore, assume that $j \geq 1$ and $F_{j-1} \in \mathcal{D}''$. We can write

$$E_{i_j - 1} \cup F_j = E_{i_j} \in \mathcal{D}''$$
$$E_{i_j - 1} \cap F_j = F_{j-1} \in \mathcal{D}''$$

where $F_{j-1} \in \mathcal{D}''$ by induction. Moreover, since $E_{i_j - 1}, F_j \in \mathcal{D}'$, we have

$$\mathsf{rk}_M(E_{i_j - 1}) + \mathsf{rk}_M(F_j) = \mathsf{rk}_M(E_{i_j}) + \mathsf{rk}_M(F_{j-1})$$

since $\mathsf{rk}_M(\cdot)$ is modular on $\mathcal{D}'$. Therefore, since $\mathcal{D}''$ is initial, we have $F_j \in \mathcal{D}''$. This completes the induction. Now, by taking $j = m$, we conclude $F \in \mathcal{D}''$. $\square$

Proposition 2.4.27. *We have $\mathcal{D} = \mathcal{D}_{\sigma_{\mathcal{D}}}$.*

Proof. Let $\mathcal{D}$ be an initial sublattice of $M|S$ for some $S \subseteq E$. Let $\mathscr{C} : \emptyset \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_k = S$ be a maximal chain of $\mathcal{D}$. By Proposition 2.4.26, we are done if we can show that we can take $\omega \in \mathsf{relint}(\mathsf{cone}(\mathbf{e}_{E_1}, \ldots, \mathbf{e}_{E_k}))$ so that M^ω is loop-free, and that $\mathscr{C}$ is a maximal chain of

$$\mathcal{D}_{\sigma_{\mathcal{D}}} = \{F = S_1 \cup \cdots \cup S_m : \text{each } S_i \in \Pi,\ \mathsf{rk}_{M^\omega}(F) = \mathsf{rk}_M(F)\}$$

where Π consists of the ground sets of the connected components of M^ω.

Let $\omega \in \mathsf{relint}(\mathsf{cone}(\mathbf{e}_{E_1}, \ldots, \mathbf{e}_{E_k}))$. That M^ω is loop-free follows from Propositions 2.4.15 and 2.4.19. We also have that each $E_i \in \mathcal{D}_{\sigma_{\mathcal{D}}}$ by Proposition 2.4.24, so that $\mathscr{C}$ is a chain of $\mathcal{D}_{\sigma_{\mathcal{D}}}$. It therefore remains to show that $\mathscr{C}$ is maximal.

For $\mathscr{C}$ to not be a maximal chain of $\mathcal{D}_{\sigma_{\mathcal{D}}}$ means that there exists some summand of

$$M^\omega = \bigoplus_{i=1}^{k} (M|E_i)/E_{i-1}$$

which is disconnected. Suppose $M' := (M|E_i)/E_{i-1} = M'_1 \oplus M'_2$, and let E'_1, E'_2 denote the nonempty ground sets of M'_1, M'_2, respectively. Let

$$F_1 := E_{i-1} \cup E'_1$$
$$F_2 := E_{i-1} \cup E'_2.$$

Then $F_1 \cup F_2 = E_i \in \mathcal{D}$ and $F_1 \cap F_2 = E_{i-1} \in \mathcal{D}$. Moreover, by Proposition 2.4.20, F_1, F_2 form a modular pair in $\mathcal{L}_{M|S}$. Since $\mathcal{D}$ is initial, we conclude that $F_1, F_2 \in \mathcal{D}$. But this contradicts maximality of $\mathscr{C}$ in $\mathcal{D}$. $\square$

<u>An extreme ray description of the cones in the outer normal fan of $P_M^{\pm}$</u>

The final step in the proof of Theorem 2.4.1 is to establish the extreme ray description 2.1 of the cones $\sigma(X, \mathcal{D})$. We start by describing the rays of the outer normal fan of the polytope $P_M^{\pm}$, which is full-dimensional in $\mathbf{R}^E$ since M is loop-free. This task has essentially been carried out by Edmonds:

Proposition 2.4.28. *[37, Theorem 40.5] A nonzero vector $\rho \in \mathbf{R}^E$ is an extreme ray of the normal fan of $P_M^{\pm}$ if and only if ρ is some nonzero multiple of some lattice point $\{-1,0,1\}^E$ such that $\mathsf{supp}(\rho)$ equals a connected flat of M.*

Proof. Let $\rho \in \mathbf{R}^E$ be a nonzero vector that is an extreme ray of the normal fan of $P_M^{\pm}$. The one-dimensional cone $\mathbf{R}_{\geq 0} \cdot \rho$ is a cone in the outer normal fan of $P_M^{\pm}$, so by Lemma 2.4.11 it corresponds to some initial sublattice $\mathcal{D}$ of $\mathcal{L}_{M|S}$ for some $S \subseteq E$. Now, the dimension of the cone $\mathbf{R}_{\geq 0} \cdot \rho$, which is equal to 1, is an upper bound on the length of a maximal flag of $\mathcal{D}$ by the description of the bijection of Lemma 2.4.11. Hence $\mathcal{D} = \{\emptyset, S\}$, and so, by this description, we see that (up to positive scaling) $\rho \in \{-1,0,1\}^E$ with $\mathsf{supp}(\rho) = S$. To see that S is a flat of M, note that if $f \in \mathsf{cl}_M(S) \setminus S$, then the face τ_ρ of $P_M^{\pm}$ maximized by ρ lies in the hyperplane $x_f = 0$. This is because every basis B of M such that $B \cap S$ is a basis of $M|S$ must satisfy

$$B \cap (\mathsf{cl}_M(S) \setminus S) = \emptyset.$$

It follows that the face of $P_M^{\pm}$ maximized by $\rho + \varepsilon \mathbf{e}_f$ would contain τ_ρ for sufficiently small $\varepsilon > 0$. But this cannot happen since τ_ρ is already a facet of $P_M^{\pm}$. To see that S is connected, note that if S is equal to the disjoint union $S_1 \cup S_2$ where S_1, S_2 are flats of M such that $\mathsf{rk}_M(S_1) + \mathsf{rk}_M(S_2) = \mathsf{rk}_M(S)$, then since $\mathcal{D}$ is initial we would have $S_1, S_2 \in \mathcal{D}$ as well. This contradicts $\mathcal{D} = \{\emptyset, S\}$.

Conversely, suppose that $\rho \in \{-1,0,1\}^E$ is nonzero and has the support of a connected flat S. We exhibit $|E| - 1$ linearly independent vectors each of which is parallel to some line segment inside the face τ_ρ dual to ρ in $P_M^{\pm}$. If $f \in E \setminus S$, then because S is a flat of M, M/S is loop-free, and therefore there must exist a vertex v of τ_ρ such that $v_f = \pm 1$. Furthermore, since $\rho_f = 0$, we have that $v - 2v_f\mathbf{e}_f$ is also a vertex of τ_ρ. We conclude that $\mathbf{e}_f$ is parallel to some line segment inside τ_ρ. Note that at this point we are done if $|S| = 1$, so assume $|S| \geq 2$. Fix some $e \in S$ and let $f \in S$ be distinct from e. We show that $\mathbf{e}_e - \mathbf{e}_f$ is parallel to some line segment inside τ_ρ. Since S is connected, there exists a circuit C of M such that $\{e, f\} \subseteq C \subseteq S$. This further implies there exists two bases B, B' of M such that $B \cap S$ and $B' \cap S$ are bases of $M|S$, and $B' = B \cup e \setminus f$. It follows that we can find two vertices $v, v' \in \tau_\rho$ such that $v - v'$ is parallel to $\mathbf{e}_e - \mathbf{e}_f$. Altogether, this produces $|E| - 1$ linearly independent vectors parallel to line segments inside τ_ρ. □

We now explain the extreme rays statement of Theorem 2.4.1.

Proof of the second part of Theorem 2.4.1. Fix a cone σ in the normal fan of $P_M^{\pm}$, and let $\omega \in \mathsf{relint}(\sigma)$ be a minimal-support element of $\mathsf{relint}(\sigma)$. Let $S = \mathsf{supp}(\omega)$, and let $\mathcal{D}$ be the initial sublattice of $\mathcal{L}_{M|S}$ corresponding to σ. We show that ρ is an extreme ray of σ if and only if $\rho \in \{-1, 0, 1\}^E$ has support equal to a connected flat $\mathsf{cl}_M(F)$ for some $F \in \mathcal{D}$ such that ρ and ω agree in sign on their common support. As before, we denote the face of $P_M^{\pm}$ maximized by ρ, ω by τ_ρ, τ_ω, respectively.

($\Rightarrow$) If ρ is an extreme ray of σ, then it is an extreme ray of $P_M^{\pm}$, so that we may assume (by the above Proposition 2.4.28) that $\rho \in \{-1, 0, 1\}^E$ and $\mathsf{supp}(\rho) = G$ for some connected flat G of M. Enumerating the ground sets of the connected components of M^ω as $S_1, S_2, \ldots, S_k$, and letting $F = G \cap S$, our goal is to show three things:

- The set F is the union of the S_i's,
- We have $\mathsf{rk}_{M^\omega}(F) = \mathsf{rk}_M(F)$, and
- We have $G = \mathsf{cl}_M(F)$.

Note that the combination of the first two conditions is equivalent to $F \in \mathcal{D}$, by the description of the map $\sigma \mapsto \mathcal{D}_\sigma$ in Section 2.4.3.

First we show that, for each S_i, either F contains S_i or is disjoint from S_i. Let $e, f \in S_i$ which, for this purpose, we assume to be distinct. Since $M^\omega | S_i$ is a connected component of M^ω, there exists a circuit C of M^ω containing both e and f. It follows that there exists bases B, B' of M^ω such that $B' = B \cup f \smallsetminus e$. Now, by Proposition 2.4.7 (3), there is a vertex v of τ_ω such that $B = \mathsf{supp}(v) \cap S$, and a vertex v' of τ_ω such that $B' = \mathsf{supp}(v') \cap S$. Because $\tau_\omega \subseteq \tau_\rho$, we have that ρ attains the same objective value at both v and v', and therefore we conclude

$$|B \cap F| = \langle \rho, v \rangle = \langle \rho, v' \rangle = |B' \cap F| = |(B \cup f \smallsetminus e) \cap F| \,.$$

In particular, it is not the case that F contains e but not f. By symmetry, it is also not the case that F contains f but not e. Repeating this argument for every pair (e, f') where $f' \in S_i \smallsetminus e$ shows that F either contains S_i or is disjoint from S_i.

To see that $\mathsf{rk}_{M^\omega}(F) = \mathsf{rk}_M(F)$ and $G = \mathsf{cl}_M(F)$, choose any vertex v of τ_ω. Since $\tau_\omega \subseteq \tau_\rho$, we have $\langle \rho, v \rangle = |B \cap G| = \mathsf{rk}_M(G)$, where $B = \mathsf{supp}(v)$ is a basis of M. Since $v \in \tau_\omega$, we also have $B \cap S$ is a basis of M^ω by Proposition 2.4.7 (3). We just showed F is a union of the ground sets of the connected components of M^ω, and this implies $\mathsf{rk}_{M^\omega}(F) = |(B \cap S) \cap F| = |B \cap F|$. We next show that B does not intersect $G \smallsetminus S$. If there were some $f \in B \cap (G \smallsetminus S)$, then since f is not in $S = \mathsf{supp}(\omega)$ we must have $v' := v - 2v_f \mathbf{e}_f$ is also a vertex of τ_ω, and since $\tau_\omega \subseteq \tau_\rho$ this would imply

$$|B \cap F| - 2 = \langle \rho, v' \rangle = \langle \rho, v \rangle = |B \cap F| \,,$$

which is a contradiction. So B is indeed disjoint from $G \smallsetminus S$, and therefore $B \cap F =$

$B \cap G$. Putting this all together, we get

$$\mathsf{rk}_M(F) \geq \mathsf{rk}_{M^\omega}(F) = |B \cap F| = |B \cap G| = \mathsf{rk}_M(G) \geq \mathsf{rk}_M(F)$$

so that equality holds throughout, and in particular $\mathsf{rk}_M(F) = \mathsf{rk}_{M^\omega}(F)$ and $G = \mathsf{cl}_M(F)$.

($\Leftarrow$) Assume $\rho \in \{-1, 0, 1\}^E$ has support equal to a connected flat $G := \mathsf{cl}_M(F)$ for some $F \in \mathcal{D}$, so that ρ agrees in sign with ω on the mutual support of ρ and ω. We show that ρ is an extreme ray of σ. To do this, it suffices to show that ρ maximizes every vertex v in τ_ω. So let v be a vertex of τ_ω, and let $B = \mathsf{supp}(v)$ be the corresponding basis of M. Since $v \in \tau_\omega$, we have $B \cap S$ is a basis of M^ω, and since $F \in \mathcal{D}$ we have that F is a union of the ground sets of the connected components of M^ω, which implies $B \cap F$ is a basis of $M^\omega|F$. Furthermore, again since $F \in \mathcal{D}$, we have $\mathsf{rk}_{M^\omega}(F) = \mathsf{rk}_M(F)$. Therefore, we get

$$\langle \rho, v \rangle = |B \cap G| \geq |B \cap F| = \mathsf{rk}_{M^\omega}(F) = \mathsf{rk}_M(F) = \mathsf{rk}_M(G),$$

and since $\langle \rho, w \rangle \leq \mathsf{rk}_M(G)$ for every vertex w of $P_M^\pm$, we conclude that the vertices of τ_ω attain the maximum possible objective value of ρ among all vertices of $P_M^\pm$. $\square$

Proofs of the main results

The content of this section expands on and proves the main results of this paper, Theorem 2.3.4 and Corollary 2.3.5.

The main theorem

Let $\mathcal{M}$ be a loop-free oriented matroid, let $\omega \in \mathbf{R}^E$ with flag

$$\emptyset = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_k = S$$

as in Definition 2.4.6, and let $A = \{f \in E : \omega_f < 0\}$. Define the oriented matroid

$$\mathcal{M}^\omega := {}_{-A}\left(\bigoplus_{i=1}^{k} (\mathcal{M}|E_i)/E_{i-1}\right).$$

Note that if M is the underlying matroid of an oriented matroid $\mathcal{M}$, then M^ω is the underlying matroid of $\mathcal{M}^\omega$.

Let τ_ω denote the face of $P_M^{\pm}$ that is maximized by ω. By the *loop-free part of* $\mathcal{M}$, we mean the oriented matroid $\mathcal{M} \setminus L$ where L is the set of loops of $\mathcal{M}$. In the oriented setting, Proposition 2.4.9 takes the following form:

Proposition 2.5.1. *For two vectors $\omega, \omega' \in \mathbf{R}^E$, we have $\tau_\omega = \tau_{\omega'}$ if and only if the loop-free part of $\mathcal{M}^\omega$ equals the loop-free part of $\mathcal{M}^{\omega'}$.*

Recall that an oriented matroid is *totally cyclic* if every element of the ground set is contained in a positive circuit.

Proposition 2.5.2. *[23, Theorem 3.4] The support of $\Sigma_\mathcal{M}$ is given by*

$$|\Sigma_\mathcal{M}| = \left\{\omega \in \mathbf{R}^E : \mathcal{M}^\omega \textit{ is totally cyclic}\right\}.$$

Proof. We have that $\mathcal{M}^\omega$ is totally cyclic if and only if each summand of $\mathcal{M}^\omega$ is. The positive circuits of a summand

$${}_{-A\cap(E_i \setminus E_{i-1})}((\mathcal{M}|E_i)/E_{i-1})$$

of $\mathcal{M}^\omega$ are the inclusionwise minimal sign vectors of the form $C \setminus E_{i-1}$, where C is a signed circuit of $\mathcal{M}|E_i$, and C agrees in sign with ω on $C \setminus E_{i-1}$ (Prop 3.3.2 red book). Hence, every element in $E_i \setminus E_{i-1}$ is contained in a positive circuit of this summand if and only if there exists a vector X_i of $\mathcal{M}|E_i$ such that $X_i(e) = \mathsf{sign}(\omega_e)$ for each $e \in E_i \setminus E_{i-1}$. Here we are using the fact that every vector is a conformal composition of circuits. This is the same as saying that there exists a flag of vectors $\mathbf{0} < X_1 < X_2 < \cdots < X_k$ of $\mathcal{M}$ such that

$$\omega = \lambda_1 \mathbf{e}_{X_1} + \lambda_2 \mathbf{e}_{X_2} + \cdots + \lambda_k \mathbf{e}_{X_k}$$

for some $\lambda_1, \ldots, \lambda_k > 0$. But this is precisely the statement that $\omega \in |\Sigma_{\mathcal{M}}|$, by definition of the fine subdivision of $\Sigma_{\mathcal{M}}$. $\square$

Since an oriented matroid $\mathcal{M}$ is totally cyclic if and only if the loop-free part of $\mathcal{M}$ is totally cyclic, we get the following corollary of Propositions 2.5.2 and 2.5.1:

Corollary 2.5.3. *The support of $\Sigma_{\mathcal{M}}$ is subdivided by cones in the outer normal fan of the signed matroid polytope $P_M^{\pm}$.*

We can make this statement more precise in terms of the pairs $(X, \mathcal{D})$ of Theorem 2.4.1:

Corollary 2.5.4. *This subdivision of $\Sigma_{\mathcal{M}}$ is given by*

$$\Sigma_{\mathcal{M}} = \{\sigma(X, \mathcal{D}) : X \cap F \text{ is a vector of } \mathcal{M} \text{ for all } F \in \mathcal{D}\}.$$

Proof. First, let $\sigma \in \Sigma_{\mathcal{M}}$. Then by Theorem 2.4.1, $\sigma = \sigma(X, \mathcal{D})$ for some pair $X, \mathcal{D}$. Now let $F \in \mathcal{D}$, and let $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$ be a maximal chain of $\mathcal{D}$ so that $F = F_j$ for some j. We show that $X \cap F_i$ is a vector of $\mathcal{M}$ for all $i = 1, 2, \ldots, k$. By Theorem 2.4.1,

$$\omega := \mathbf{e}_{X \cap F_1} + \mathbf{e}_{X \cap F_2} + \cdots + \mathbf{e}_{X \cap F_k}$$

lies in the relative interior of $\sigma(X, \mathcal{D})$. Since $\sigma(X, \mathcal{D}) \in \Sigma_{\mathcal{M}}$, we have $\omega \in |\Sigma_{\mathcal{M}}|$. Therefore, by the fine subdivision of $\Sigma_{\mathcal{M}}$, we have

$$\omega \in \mathsf{relint}(\mathsf{cone}(\mathbf{e}_{X_1}, \mathbf{e}_{X_2}, \ldots, \mathbf{e}_{X_m}))$$

for some flag of conformal vectors $X_1 < X_2 < \cdots < X_m$ of $\mathcal{M}$. Now this flag of conformal vectors can be recovered from ω, which implies that $k = m$ and $X_i = X \cap F_i$ for all $i = 1, 2, \ldots, k$. We conclude $X \cap F = X \cap F_j = X_j$ is a vector of $\mathcal{M}$.

Conversely, suppose $\sigma(X, \mathcal{D})$ has the property that $X \cap F$ is a vector for all $F \in \mathcal{D}$. Choose any maximal chain $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$ of $\mathcal{D}$, and let

$$\omega = \mathbf{e}_{X\cap F_1} + \mathbf{e}_{X\cap F_2} + \cdots + \mathbf{e}_{X\cap F_k}$$

so that by Theorem 2.4.1, $\omega \in \mathsf{relint}(\sigma(X, \mathcal{D}))$. Then we have

$$\omega \in \mathsf{relint}(\mathsf{cone}(\mathbf{e}_{X\cap F_1}, \mathbf{e}_{X\cap F_2}, \ldots, \mathbf{e}_{X\cap F_k})),$$

which is a cone in the fine subdivision of $\Sigma_{\mathcal{M}}$ since

$$X \cap F_1 < X \cap F_2 < \cdots < X \cap F_k$$

is a flag of conformal vectors of $\mathcal{M}$ by assumption. It follows that the relative interior of $\sigma(X, \mathcal{D})$ intersects $|\Sigma_{\mathcal{M}}|$, and this implies $\sigma(X, \mathcal{D}) \in \Sigma_{\mathcal{M}}$ by Corollary 2.5.4. $\square$

The uniform case

In this section, we assume that $\mathcal{M}$ is a loop-free *uniform* oriented matroid on the ground set E, with underlying matroid M.

We begin by showing Corollary 2.5.4 can be made more precise in the uniform setting. Write as shorthand

$$\sigma(X) := \sigma(X, \{\emptyset, S\})$$

where S is the support of the sign vector X. As we shall see, $\mathcal{D} = \{\emptyset, S\}$ is in fact an initial sublattice of $\mathcal{L}_{M|S}$ if X is a vector of $\mathcal{M}$.

Proposition 2.5.5. *The coarse subdivision of $\Sigma_{\mathcal{M}}$ is given by*

$$\Sigma_{\mathcal{M}} = \{\sigma(X) : X \text{ is a vector of } \mathcal{M}\}.$$

Proof. Let X be a vector of $\mathcal{M}$ with support S, and let $\mathcal{D}$ be an initial sublattice of $\mathcal{L}_{M|S}$. By Corollary 2.5.4, to prove this proposition it suffices to show that $X \cap F$ is a vector for every $F \in \mathcal{D}$ if and only if $\mathcal{D} = \{\emptyset, S\}$. The "if" direction follows from the fact that $M|S$ is connected and so S is a connected flat of $M|S$, so that $\mathcal{D}$ is not missing any other flats of $M|S$. For the "only if" direction, suppose $F \in \mathcal{D}$ is the support of a nonzero vector of $\mathcal{M}$. Then F is a cyclic flat[2] of $M|S$. But, since $M|S$ is uniform, the only possible cyclic flat of $M|S$ is S itself. Hence, $F = S$. $\square$

Corollary 2.5.6. *The poset (with respect to inclusion) of nonzero cones in the coarse subdivision of $\Sigma_{\mathcal{M}}$ is anti-isomorphic to the poset of nonzero vectors of $\mathcal{M}$.*

Proof. The previous Proposition 2.5.5 defines a bijective map between the nonzero vectors of $\mathcal{M}$ and the nonzero cones of $\Sigma_{\mathcal{M}}$. It therefore remains to show that this map is order-reversing. Let X, Y be nonzero vectors of $\mathcal{M}$ with supports S, T, respectively. Since S, T are both unions of circuits, we have that they are both dependent in M. Moreover, since M is uniform, we further have that both S and T contain a basis of M. In particular we have $\mathsf{cl}_M(S) = \mathsf{cl}_M(T) = E$. Therefore, the descriptions of the cones $\sigma(X)$ and $\sigma(Y)$ given by (2.1) simplify to

$$\sigma(X) = \mathsf{cone}\left(\rho \in \mathbf{R}^E \ : \ \begin{array}{ll} \rho_e = X_e & \text{if } e \in S \\ \rho_e \in \{-1, 1\} & \text{if } e \in E \setminus S \end{array}\right)$$

and

$$\sigma(Y) = \mathsf{cone}\left(\rho \in \mathbf{R}^E \ : \ \begin{array}{ll} \rho_e = Y_e & \text{if } e \in T \\ \rho_e \in \{-1, 1\} & \text{if } e \in E \setminus T \end{array}\right).$$

[2]A *cyclic flat* is a flat that is also a union of circuits.

From these descriptions we see that $X \leq Y$ if and only if $\sigma(X) \supseteq \sigma(Y)$. $\square$

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CHAPTER 3

A CHIROTOPE-BASED PROOF OF THE BOHNE-DRESS THEOREM

Introduction

The fundamental theorem on tilings of zonotopes by zonotopes is surely the Bohne-Dress Theorem, which states that zonotopal tilings of a fixed zonotope can be understood purely combinatorially using the theory of oriented matroids:

Theorem 3.1.1 (The Bohne-Dress theorem). *Let $\mathcal{A} = (v_1, v_2, \ldots, v_n)$ be a vector configuration of vectors in $\mathbf{R}^r$, let $\mathcal{M}$ be the oriented matroid associated to this vector configuration, and let*

$$\mathcal{Z} = \sum_{i=1}^{n} [-v_i, v_i]$$

be the zonotope associated to this vector configuration. Then there exists a 1-1 correspondence between the zonotopal tilings of $\mathcal{Z}$ and single-element liftings of $\mathcal{M}$. Here, the tiles of the zonotopal tilings are assumed to be of the form

$$\mathcal{Z}_J = \sum_{i \in J} [-v_i, v_i]$$

for some $J \subseteq \{1, 2, \ldots, n\}$.

Since the original proof appeared in Jochen Bohne's PhD thesis [38], several other proofs have appeared in the literature. A proof by Ziegler and Richter-Gebert [**zieglerrichtergebert2001**] uses McMullen's formula for the volume a zonotope to show that every single-element lifting of $\mathcal{M}$ contributes a zonotopal tiling. Huber, Rambau, and Santos [39] used the Cayley trick to show that the poset of zonotopal tilings, ordered by refinement, is isomorphic to the poset of subdivisions of the Lawrence polytope associated to the vector configuration $\mathcal{A}$. This is the polytope which is the convex hull of the columns of the matrix

$$\begin{pmatrix} A & \mathbf{0} \\ I & I \end{pmatrix},$$

where A is the $r \times n$ matrix $A = \begin{pmatrix} v_1 \; v_2 \; \cdots \; v_n \end{pmatrix}$.

The goal of this paper is to understand one direction of theorem from a topological point of view, namely, that every single element lifting of a realizable oriented matroid can be represented as a zonotopal tiling. Specifically, we show how this direction of the Bohne-Dress theorem follows from the following lemma about continuous maps between spheres: If S^k has a triangulation $\mathcal{T}$, and if a map $f : S^k \to S^k$ is continuous, has degree 1, and if the restriction $f|_\sigma : \sigma \to f(\sigma)$ is an orientation-preserving homeomorphism for each $\sigma \in \mathcal{T}$, then f itself is a homeomorphism. A crucial ingredient here is the Topological Representation Theorem of Folkman and Lawrence [27, Theorem 5.2.1], which asserts that $\Delta_{\mathcal{M}}$, the order complex of the poset of nonzero covectors of an oriented matroid $\mathcal{M}$, is homeomorphic to a sphere.

The outline of this paper is as follows: After reviewing some notation and basic definitions, we state the particular formulation of the Bohne-Dress theorem we are interested in proving. Next, we give an alternative interpretation of the chirotope of an oriented matroid, one which makes it clear how to consistently orient the simplices of $\Delta_{\mathcal{M}}$ to match the orientation of $\|\Delta_{\mathcal{M}}\| \simeq S^{r-1}$. After this, we state and prove our version of the Bohne-Dress theorem using the lemma about maps between spheres mentioned above. The second-to-last section is then dedicated to proving the lemma, and can be considered as an appendix. Finally, in the last section, we speculate on generalizations of the Bohne-Dress theorem to settings beyond the realizable case.

Statement of the main result

Before stating the main result, we review some notation and definitions used in this paper. We assume that the reader is familiar with oriented matroids. For an introduction to oriented matroids, we refer the reader to [27] or [28]. We define

$$\mathbf{3} := \{-1, 0, +1\}$$
$$\mathbf{2} := \{-1, +1\}.$$

For oriented matroids $\mathcal{M}, \mathcal{N}$, we write M, N to denote their underlying unoriented matroids.

Let $\mathcal{M}$ be an oriented matroid of rank r on the ground set E. Let $\Delta_{\mathcal{M}}$ denote the order complex of the poset of nonzero covectors of M:

$$\Delta_{\mathcal{M}} := \Delta(\mathcal{L}(\mathcal{M}) - \mathbf{0}).$$

Here $\mathcal{V}^*(\mathcal{M})$ denotes the set of covectors of $\mathcal{M}$. We identify $\Delta_{\mathcal{M}}$ with the following geometric realization of $\Delta_{\mathcal{M}}$ in $\mathbf{R}^E$:

$$\Delta_{\mathcal{M}} := \left\{ \mathsf{conv}\left(\mathbf{e}_{X_1}, \ldots, \mathbf{e}_{X_k}\right) \ : \ X_1 < X_2 < \cdots < X_k, \text{ each } X_i \in \mathcal{V}^*(M) \right\}.$$

Here, given a sign vector $X \in \mathbf{3}^E$, we define

$$\mathbf{e}_X := \left(\sum_{f \in X^+} \mathbf{e}_f \right) - \left(\sum_{f \in X^-} \mathbf{e}_f \right) \in \mathbf{R}^E.$$

Given two matroids $\mathcal{M}$, $\mathcal{N}$ on the ground set E of the same rank r, we say that *there is a weak map* $\mathcal{M} \rightsquigarrow \mathcal{N}$ if, up to a global sign change, we have

$$\chi_{\mathcal{M}} \geq \chi_{\mathcal{N}},$$

where $a \geq b$ means b is obtained from a by setting some entries to zero. Here $\chi_{\mathcal{M}}$ and $\chi_{\mathcal{N}}$ are chirotopes of $\mathcal{M}, \mathcal{N}$, respectively.

For our purposes, the following stronger notion of a weak map will be important for us:

Definition 3.2.1. A weak map $\mathcal{M} \rightsquigarrow \mathcal{N}$ is *rank-preserving on flats* if

$$\mathsf{rk}(M|F) = \mathsf{rk}(N|F)$$

for all flats F of M.

Note that there are weak maps $\mathcal{M} \rightsquigarrow \mathcal{N}$ which are not rank-preserving on flats (Figure .), even when $\mathcal{M}$ and $\mathcal{N}$ have the same rank.

Definition 3.2.2. Let $\pi : \mathbf{R}^E \to \mathbf{R}^r$ be a surjective linear map. The *oriented matroid corresponding to* π is the oriented matroid $\mathcal{N}$ whose chirotope is defined by

$$\chi_{\mathcal{N}}(b_1, \ldots, b_r) = \mathsf{sign} \det (v_{b_1}, \ldots, v_{b_r}), \quad v_f := \pi(\mathbf{e}_f)$$

for all r-tuples $(b_1, \ldots, b_r) \in E^r$.

We are now ready to state our version of the Bohne-Dress theorem.

Theorem 3.2.3. *Let $\mathcal{M} \rightsquigarrow \mathcal{N}$ be a weak map pair such that $\mathcal{N}$ is the oriented matroid of a surjective linear map $\pi : \mathbf{R}^E \to \mathbf{R}^r$. Assume $\mathcal{N}$ is loop-free. Let $\mathsf{bd}\,\mathcal{Z}$ denote the boundary of the zonotope $\mathcal{Z} := \pi([-1,1]^E)$. Then π restricts to a homeomorphism $\pi_{\mathcal{M}} : \|\Delta_{\mathcal{M}}\| \to \mathsf{bd}\,\mathcal{Z}$ if and only if $\mathcal{M} \rightsquigarrow \mathcal{N}$ is rank-preserving on flats.*

The chirotope, revisited

In this section we reinterpret the chirotope of an oriented matroid $\mathcal{M}$ in terms of flags of conformal covectors. Consider an arrangement of pseudospheres representing $\mathcal{M}$

inside the sphere S^{r-1}. The pieces cut out by the pseudospheres fit together to form a cell complex of S^{r-1}, and, by taking the first barycentric subdivision of this complex, we obtain a triangulation of S^{r-1}. There is, furthermore, a natural ordering on the vertices of each simplex in this triangulation: the vertices of each simplex correspond to the covectors of a maximal flag of conformal covectors, and we can order these by containment. The main observation of this section is that the chirotope assigns a $+$ or a $-$ to each maximal simplex in this triangulation, according to whether or not the simplex (with its natural ordering of vertices) agrees or disagrees with a fixed orientation of S^{r-1}.

Signed ordered bases

A *signed, ordered basis* $B = (s_1b_1, s_2b_2, \ldots, s_rb_r)$ is an ordered r-tuple such that each $s_i \in \{-1, 1\}$ and $\{b_1, \ldots, b_r\}$ is a basis of $\mathcal{M}$. We will shorten this term to "s.o. basis" for brevity. The first statement we make is that a s.o. basis uniquely determines a maximal flag of covectors of $\mathcal{M}$.

Proposition 3.3.1. *Let $B = (s_1b_1, \ldots, s_rb_r)$ be a s.o. basis. Then there exists a unique flag of covectors $\mathscr{F} : \mathbf{0} = X_0 < X_1 < X_2 < \cdots < X_r$ such that $b_i \in \mathsf{supp}(X_i) \smallsetminus \mathsf{supp}(X_{i-1})$ and $X_i(b_i) = s_i$ for all $i = 1, 2, \ldots, r$.*

We will denote the flag associated to a s.o. basis B as $\mathscr{F}_B$.

Proof. Assume $r \geq 1$. Let X_1 be the cocircuit complementary to the flat F_1 spanned by $b_2, \ldots, b_r$, so that E is the disjoint union $\mathsf{supp}(X_1) \cup F_1$. Then X_1 is determined up to sign, and so we further specify that $X_1(b_1) = s_1$. By induction, there is a unique flag of covectors

$$\mathscr{F}' : \mathbf{0} = X'_0 < X'_2 < \cdots < X'_r$$

of $\mathcal{M}|F_1$ associated to the s.o. basis $(s_2b_2, \ldots, s_rb_r)$. From this flag we construct the flag $\mathscr{F} : \mathbf{0} = X_0 < X_1 < X_2 < \cdots < X_r$ by setting

$$X_i(f) = \begin{cases} X_1(f), & f \in \mathsf{supp}(X_1) \\ X'_r(f), & f \in F_1 \end{cases}$$

for each $i = 1, 2, \ldots, r$. This procedure determines $\mathscr{F}$ uniquely, and any $\mathscr{F}$ satisfying the conclusions of Proposition 3.3.1 can be recovered using this procedure given $B = (s_1 b_1, \ldots, s_r b_r)$. □

A chirotope χ of $\mathcal{M}$ is an alternating function on the set of ordered bases of M, and extends naturally to an alternating function on the set of signed ordered bases as follows:

$$\chi\,(s_1 b_1, s_2 b_2, \ldots, s_r b_r) := s_1 s_2 \cdots s_r\; \chi(b_1, b_2, \ldots, b_r).$$

Proposition 3.3.2. *Let B, B' be two s.o. bases such that $\mathscr{F}_B = \mathscr{F}_{B'}$. Then $\chi(B) = \chi(B')$. In particular, $\chi(B)$ depends only on the flag of covectors determined by B.*

Proof. This statement is obvious when $r = 1$, so assume $r \geq 2$. Let $\mathscr{F} : \mathbf{0} = X_0 < X_1 < \cdots < X_r$, and let $B = (s_1 b_1, \ldots, s_r b_r)$, $C = (t_1 c_1, \ldots, t_r c_r)$ be two s.o. bases such that $\mathscr{F}_B = \mathscr{F}_C = \mathscr{F}$. Let $F_1 = E \setminus X_1$. A chirotope χ_{F_1} for the restriction $M|F_1$ is given by

$$\chi_{F_1}(x_2, \ldots, x_r) = \chi(b_1, x_2, \ldots, x_r), \quad x_2, \ldots, x_r \in F_1.$$

Let $\mathscr{F}' : \mathbf{0} = X_0 \cap F_1 < X_2 \cap F_1 < \cdots < X_r \cap F_1$ be a flag of covectors $M|F_1$, where $X_i|F_1$ is the covector of $M|F_1$ satisfying

$$(X_i \cap F_1)^+ = X_i^+ \cap F_1$$

$$(X_i \cap F_1)^0 = X_i^0 \cap F_1$$

$$(X_i \cap F_1)^- = X_i^- \cap F_1$$

for all $i = 1, 2, \ldots, r$. Then $\mathscr{F}_{B'} = \mathscr{F}_{C'} = \mathscr{F}'$, where $B' = (s_2 b_2, \ldots, s_r b_r)$, $C' = (t_2 c_2, \ldots, t_r c_r)$ are s.o. bases of $M|F_1$, by the procedure outlined in the proof of Proposition 3.3.1. By induction, then, we have $\chi_{F_1}(B') = \chi_{F_1}(C')$. Now,

$$\begin{aligned}
\chi(B) &= \chi(s_1 b_1, s_2 b_2 \ldots, s_r b_r) \\
&= s_1 \cdot \chi_{F_1}(B') \\
&= s_1 \cdot \chi_{F_1}(C') \\
&= s_1 \cdot \chi(b_1, t_2 c_2, \ldots, t_r c_r) \\
&= s_1 \cdot t_2 \cdots t_r \cdot \chi(b_1, c_2, \ldots, c_r) \\
&= s_1 \cdot t_2 \cdots t_r \cdot s_1 t_1 \cdot \chi(c_1, c_2, \ldots, c_r) \\
&= \chi(t_1 c_1, \ldots, t_r c_r) \\
&= \chi(C).
\end{aligned}$$

Here the third-to-last equality holds by the dual pivoting property [27, p. 125]. $\square$

Orienting simplices using the chirotope

Given a maximal flag $\mathscr{F} : \mathbf{0} = X_0 < X_1 < \cdots < X_r$ of covectors, define

$$\chi(\mathscr{F}) := \chi(B)$$

where B is any s.o. basis B such that $\mathscr{F}_B = \mathscr{F}$. Note that such a B always exists; one can take $B = (X_1(b_1)b_1, X_2(b_2)b_2, \ldots, X_r(b_r)b_r)$, where each b_i is chosen arbitrarily from $\mathsf{supp}(X_i) \smallsetminus \mathsf{supp}(X_{i-1})$. Proposition 3.3.2 implies that this definition is well-defined.

The next goal is to give a topological interpretation of the chirotope χ of $\mathcal{M}$. Here we make crucial use of the following fact about the topology of $\Delta_{\mathcal{M}}$:

Theorem 3.3.3 (Topological Representation Theorem [27, Theorem 5.2.1])**.** *The* *complex $\Delta_{\mathcal{M}}$ is homeomorphic to the sphere S^{r-1}.*

This theorem implies that the only nonvanishing reduced homology group of $\Delta_{\mathcal{M}}$ is $\tilde{H}_{r-1}(\Delta_{\mathcal{M}})$, which is isomorphic to $\mathbf{Z}$. Now, $\tilde{H}_{r-1}(\Delta_{\mathcal{M}})$ is spanned by simplicial maps

$$\begin{aligned} \sigma_{\mathscr{F}} : \Delta_{r-1} &\to \Delta_{\mathcal{M}} \\ \mathbf{e}_i &\mapsto \mathbf{e}_{X_i} \quad \text{for all } i = 1, 2, \ldots, r \end{aligned}$$

for each maximal flag $\mathscr{F} : \mathbf{0} = X_0 < X_1 < \cdots < X_r$ of covectors, where

$$\Delta_{r-1} := \operatorname{conv}(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r) \subset \mathbf{R}^r.$$

Recall that an *orientation* of an orientable $(r-1)$-dimensional manifold M is a cycle $\alpha \in \tilde{H}_{r-1}(M)$ which generates $\tilde{H}_{r-1}(M)$, and if $\mathcal{T}$ is a collection of maps $\sigma : \Delta_{r-1} \to M$ which determines a triangulation of M, then α can be written as a linear combination of the elements of $\mathcal{T}$ (more precisely, their images in $\tilde{H}_{r-1}(M)$) each having coefficient $+1$ or -1. In the context of $\Delta_{\mathcal{M}}$, these signs are governed by the chirotope of $\mathcal{M}$:

Proposition 3.3.4. *A generator for $\tilde{H}_{r-1}(\Delta_{\mathcal{M}})$ is given by*

$$\alpha_{\mathcal{M}} := \sum_{\mathscr{F}} \chi(\mathscr{F})[\sigma_{\mathscr{F}}],$$

where the sum runs over all maximal flags $\mathscr{F}$ of conformal covectors.

Proof. We show that $\alpha_{\mathcal{M}}$ is a cycle; the fact that $\alpha_{\mathcal{M}}$ generates $\tilde{H}_{r-1}(\Delta_{\mathcal{M}})$ will then follow from the fact that $\mathcal{T}$ is a triangulation of $\Delta_{\mathcal{M}}$. The boundary map

$\partial : C_{r-1}(\Delta_{\mathcal{M}}) \to C_{r-2}(\Delta_{\mathcal{M}})$ sends $\alpha_{\mathcal{M}}$ to

$$\begin{aligned}\partial\alpha_M &= \sum_{\mathscr{F}} \chi(\mathscr{F}) \sum_{k=1}^{r} (-1)^k [\sigma_{\mathscr{F},k}] \\ &= \sum_{\sigma \in C_{r-2}(\Delta_{\mathcal{M}})} \left(\sum_{(\mathscr{F},k):\sigma_{\mathscr{F},k}=\sigma} \chi(\mathscr{F})(-1)^k \right) [\sigma]. \end{aligned} \tag{3.1}$$

Here $\sigma_{\mathscr{F},k}$ is the map $\sigma_{\mathscr{F}}$ restricted to the facet of Δ_{r-1} not containing vertex k. To show that this is zero, it suffices to show that the inner sum of (3.1) is zero whenever it is a nonempty sum.

Denote by $\mathcal{L}$ the lattice obtained by adjoining a top element $\mathbf{1}$ to the poset $\mathcal{L}(\mathcal{M})$. Let $\sigma \in C_{r-2}(\Delta_{\mathcal{M}})$ so that the inner sum of (3.1) is nonempty. Then σ corresponds to a flag of $\mathcal{L}$ of the form

$$\mathscr{F}_\sigma : \mathbf{0} = X_0 < X_1 < \cdots < X_{k-1} < X_{k+1} < \cdots < X_r < X_{r+1} = \mathbf{1}.$$

This flag is obtained from a maximal flag $\mathscr{F} : \mathbf{0} = X_0 < X_1 < \cdots < X_r < X_{r+1} = \mathbf{1}$ of $\mathcal{L}$ by removing X_k for some $k = 1, 2, \ldots, r$. Now, $[X_{k-1}, X_{k+1}]$ is a length-2 interval in the lattice $\mathcal{L}$, and, therefore, there exist exactly two incomparable covectors X_k, X_k' strictly inside this interval. Let $\mathscr{F}$ and $\mathscr{F}'$ be the two extensions of $\mathscr{F}_\sigma$ which contain X_k and X_k', respectively. Then the inner sum of (3.1) corresponding to σ is equal to

$$\chi(\mathscr{F})(-1)^k + \chi(\mathscr{F}')(-1)^k.$$

Thus, we would like to show that $\chi(\mathscr{F}) = -\chi(\mathscr{F}')$. Let $B = (s_1b_1, \ldots, s_rb_r)$ be a s.o. basis such that $\mathscr{F} = \mathscr{F}_B$. If $k = r$, then we must have $X_r' = X_{r-1} \circ (-X_r)$, and therefore $B' := (s_1b_1, \ldots, s_{r-1}b_{r-1}, -s_rb_r)$ is an s.o. basis of $\mathscr{F}'$. It follows that

$$\chi(\mathscr{F}') = \chi(s_1b_1, \ldots, -s_rb_r) = -\chi(s_1b_1, \ldots, s_rb_r) = -\chi(\mathscr{F}).$$

Otherwise, $k < r$, and in this case $B' = (s_1 b_1, \ldots, s_{k+1} b_{k+1}, s_k b_k, \ldots, s_r b_r)$ is an s.o. basis for $\mathscr{F}'$. That is, B' is obtained from B by swapping the entries in the k and $k+1$ positions. Since χ is alternating, we obtain that

$$\chi(\mathscr{F}') = \chi(B') = -\chi(B) = -\chi(\mathscr{F})$$

in this case as well. This shows that $\alpha_{\mathcal{M}}$ is indeed a cycle. $\square$

Piecewise linear topology

In this section we state some basic notions from piecewise linear topology, and state a key lemma. A general reference is [40].

Definition 3.4.1. A pure k-dimensional *PL simplicial complex* K is a realization of an abstract simplicial complex in some Euclidean space $\mathbf{R}^n$, given by a collection of affine maps $\mathcal{T}_K = \{\sigma : \Delta_k \to K\}$ which are linearly isomorphic onto their images.

For our purposes, it will be convenient to keep track of the maps themselves, rather than just their images in K. It is not a loss of generality to assume K is embedded in some Euclidean space, since every abstract simplical complex of dimension k has a realization as a PL simplicial complex in $\mathbf{R}^{2k}$.

Definition 3.4.2. The support $\|K\|$ of a PL simplicial complex K is defined to be the union of $\mathsf{im}\sigma$ over all $\sigma \in \mathcal{T}_K$.

Definition 3.4.3. A *refinement* K' of K is a PL simplicial complex such that $\|K'\| = \|K\|$ and for all $\sigma' \in \mathcal{T}_{K'}$, we have $\mathsf{im}\sigma' \subseteq \mathsf{im}\sigma$ for some $\sigma \in \mathcal{T}_K$.

Definition 3.4.4. We say that a continuous map $f : K \to L$ is a *PL map* provided there exists a refinement K' of K and a refinement L' of L such that for every $\sigma' \in \mathcal{T}_{K'}$, the restriction $f|_{\sigma'}$ is a linear map whose image in L' is equal to $\mathsf{im}\tau$ for some $\tau \in \mathcal{T}_{L'}$.

Remark 3.4.5. If $f : K \to L$ is the restriction of some linear map $f : \mathbf{R}^n \to \mathbf{R}^r$, then $f : K \to L$ is automatically a PL map. [40, Lemma 1.9]

Definition 3.4.6. Let L be a PL simplicial complex. A point $y \in L$ is called a *regular point* if there exists exactly one simplex $\tau : \Delta_k \to L$ for which $y \in \mathsf{im}\tau$. In other words, y "does not lie on the boundary of any simplex."

Definition 3.4.7. Let $f : K \to L$ be a PL map. *A regular point of f* is a point $y \in L$ such that y is a regular point of L and x is a regular point of K for all $x \in f^{-1}(y)$.

Lemma 3.4.8. *Let K, L be PL k-spheres with triangulations $\mathcal{T}_K, \mathcal{T}_L$, respectively. Assume that $\tilde{H}_k(K)$ and $\tilde{H}_k(L)$ are generated by*

$$\sum_{\sigma \in \mathcal{T}_K} [\sigma] \quad \textit{and} \quad \sum_{\tau \in \mathcal{T}_L} [\tau],$$

respectively. Let $f : K \to L$ be a PL map. Assume that:

1. *There exists a subcomplex K_0 of K homeomorphic to S^{k-1}, so that the restriction $f : K_0 \to f(K_0)$ is a homeomorphism. If K_+, K_- denote the two closed hemispheres in K whose common boundary is K_0, then we also have*

$$f(K_+) \cap f(K_-) = f(K_0).$$

2. *The composition $f \circ \sigma : \Delta_k \to L$ is homeomorphic onto its image for each $\sigma \in \mathcal{T}_K$.*

3. *For each regular point y of f, and for each $x \in f^{-1}(y)$, the Jacobian determinant of the composition*

$$\tau^{-1} \circ f \circ \sigma : \Delta_k \to \Delta_k$$

is positive at $\sigma^{-1}(x)$, where σ, τ are the unique maps of $\mathcal{T}_K, \mathcal{T}_L$, respectively, such that $x \in \mathsf{im}\sigma$ and $y \in \mathsf{im}\tau$.

Then f is a PL homeomorphism.

Remark 3.4.9. In the above Lemma 3.4.8, note that the composition $\tau^{-1} \circ f \circ \sigma : \Delta_k \to \Delta_k$ is only defined in a neighbourhood U_0 of $\sigma^{-1}(x)$. We can compute the Jacobian determinant of an affine map $\varphi : U_0 \to \Delta_k$, where U_0 is open in Δ_k, by noting that φ is the restriction of a linear map $B : \mathbf{R}^{k+1} \to \mathbf{R}^{k+1}$. The Jacobian determinant, in this case, is simply the determinant of B.

The Bohne-Dress theorem, revisited

We are almost ready to state our proof of Theorem 3.2.3 using Lemma 3.4.8. First, however, we state and prove some facts about weak maps that are rank preserving on flats. We begin by showing that this notion of an oriented matroid map affords the following useful feature:

Proposition 3.5.1. *If a weak map $\mathcal{M} \rightsquigarrow \mathcal{N}$ is rank-preserving on flats, then it induces a weak map $\mathcal{M}|F \rightsquigarrow \mathcal{N}|F$ that is rank-preserving on flats for every flat F of M.*

Proof. Let F be a flat of M, and let $b_1, \dots, b_k$ be a basis of $N|F$. Since this weak map is rank-preserving on flats, $b_1, \dots, b_k$ is also a basis of $M|F$. Therefore, the chirotopes of $\mathcal{M}|F, \mathcal{N}|F$ are given by

$$\chi_{\mathcal{M}|F}(f_1, \dots, f_{r-k}) = \chi_{\mathcal{M}}(b_1, \dots, b_k, f_1, \dots, f_{r-k})$$
$$\chi_{\mathcal{N}|F}(f_1, \dots, f_{r-k}) = \chi_{\mathcal{N}}(b_1, \dots, b_k, f_1, \dots, f_{r-k})$$

for all $(f_1, \dots, f_{r-k}) \in E^{r-k}$. In particular, $\chi_{\mathcal{N}|F}$ is obtained from $\chi_{\mathcal{M}|F}$ by setting some entries to zero. This shows there is a weak map $\mathcal{M}|F \rightsquigarrow \mathcal{N}|F$. To see that this weak map is rank-preserving on flats, suppose G is a flat of $M|F$. Then G is also a flat of M, and hence

$$\mathsf{rk}_{M|F}(G) = \mathsf{rk}_M(G) = \mathsf{rk}_N(G) = \mathsf{rk}_{N|F}(G).$$

$\square$

Lemma 3.5.2. *Let $\mathcal{N}$ be the oriented matroid associated to a linear map $\pi : \mathbf{R}^E \to \mathbf{R}^r$, and assume there is a weak map $\mathcal{M} \rightsquigarrow \mathcal{N}$. Let $\mathscr{F} : \mathbf{0} = X_0 < X_1 < \cdots < X_r$ be a maximal flag of covectors of $\mathcal{M}$. Let $v_X := \pi(\mathbf{e}_X)$ for $X \in \mathbf{3}^E$. Then*

$$\chi_{\mathcal{M}}(\mathscr{F}) \det(v_{X_1}, \ldots, v_{X_r}) = \sum_S \chi_{\mathcal{N}}(S)^2 \left|\det(v_{b_1}, \ldots, v_{b_r})\right|,$$

where the sum is over all s.o. bases $S = (s_1 b_1, \ldots, s_r b_r)$ of $\mathscr{F}$. In particular, the vectors $v_{X_1}, \ldots, v_{X_r}$ are linearly independent if and only if there exists a basis $B = \{b_1, \ldots, b_r\}$ of N such that $b_i \in \mathsf{supp} X_i \smallsetminus \mathsf{supp} X_{i-1}$ for all $i = 1, 2, \ldots, r$.

Proof. We have

$$\begin{aligned}
\chi_{\mathcal{M}}(\mathscr{F}) \det(v_{X_1}, \ldots, v_{X_r}) &= \sum_S \chi_{\mathcal{M}}(\mathscr{F}) \det(v_{b_1}, \ldots, v_{b_r}) \\
&= \sum_S \chi_{\mathcal{M}}(S) \det(s_1 v_{b_1}, \ldots, s_1 v_{b_r}) \\
&= \sum_S \chi_{\mathcal{M}}(S) \chi_{\mathcal{N}}(S) \left|\det(v_{b_1}, \ldots, v_{b_r})\right| \\
&= \sum_S \chi_{\mathcal{N}}(S)^2 \left|\det(v_{b_1}, \ldots, v_{b_r})\right|.
\end{aligned}$$

The first equality holds by multilinearity of the determinant, the second equality holds by Lemma 3.3.2, the third equality holds by definition of $\chi_{\mathcal{N}}(S)$, and the last inequality holds by the weak map $\mathcal{M} \rightsquigarrow \mathcal{N}$. $\square$

In the case when $\mathcal{N}$ is realizable, there is a linear-algebraic characterization of when a weak map $\mathcal{M} \rightsquigarrow \mathcal{N}$ is rank preserving on flats:

Lemma 3.5.3. *Let $\mathcal{M} \rightsquigarrow \mathcal{N}$ be a weak map pair such that $\mathcal{N}$ is the oriented matroid of a surjective linear map $\pi : \mathbf{R}^E \to \mathbf{R}^r$. Then the following are equivalent:*

1. *The weak map $\mathcal{M} \rightsquigarrow \mathcal{N}$ is rank preserving on flats.*

2. *For all maximal flags of covectors $\mathscr{F} : \mathbf{0} = X_0 < X_1 < \cdots < X_r$ of $\mathcal{M}$, the vectors $v_{X_1}, \ldots, v_{X_r}$ are linearly independent.*

Proof. Since every flat of M is complementary to some covector X of $\mathcal{M}$, and every covector X of $\mathcal{M}$ is contained in a maximal flag of covectors $\mathscr{F}$, it suffices to show, for every maximal flag $\mathscr{F}$,

$$v_{X_1}, \ldots, v_{X_r} \text{ linearly independent } \iff \mathsf{rk}(M|F_i) = \mathsf{rk}(N|F_i) \text{ for all } i$$

where $F_i = E \smallsetminus X_{r-i}$ for $i = 0, 1, 2, \ldots, r$. for every such flag $\mathscr{F}$, By Lemma 3.5.2, it suffices to show that $\mathsf{rk}(M|F_i) = \mathsf{rk}(N|F_i)$ for all i if and only if there exists a basis B of N such that $b_i \in \mathsf{supp} X_i \smallsetminus \mathsf{supp} X_{i-1}$ for all $i = 1, 2, \ldots, r$. We note here the fact that $\mathsf{rk}(M|F_i) = i$ for all i.

Let $\mathscr{F} : \mathbf{0} = X_0 < X_1 < \cdots < X_r$ be a maximal flag of covectors of $\mathcal{M}$. First suppose we have such a basis B of N as above, and let $1 \le i \le r$. Then $b_{r-i+1}, \ldots, b_r$ is an independent set in $N|F_i$ for all i, and hence

$$i \le \mathsf{rk}(N|F_i) \le \mathsf{rk}(M|F_i) = i$$

where the second inequality holds by the weak map. This shows $\mathsf{rk}(M|F_i) = \mathsf{rk}(N|F_i)$. On the other hand, now suppose $\mathsf{rk}(M|F_i) = \mathsf{rk}(N|F_i)$ for all i. Then a basis $\{b_r\}$ of $N|F_1$ can be extended to a basis $\{b_r, b_{r-1}\}$ of $N|F_2$, which can in turn be extended to a basis $\{b_r, b_{r-1}, b_{r-2}\}$ of $N|F_3$, and so on, until we obtain a basis $B = \{b_1, \ldots, b_r\}$ of N so that $b_{r-i+1} \in F_i \smallsetminus F_{i-1}$ for all $1 \le i \le r$. Equivalently, $b_i \in \mathsf{supp} X_i \smallsetminus \mathsf{supp} X_{i-1}$ for all i. $\square$

For the convenience of the reader, we restate the main theorem of this paper which we now prove.

Theorem. *Let $\mathcal{M} \rightsquigarrow \mathcal{N}$ be a weak map pair such that $\mathcal{N}$ is the oriented matroid of a surjective linear map $\pi : \mathbf{R}^E \to \mathbf{R}^r$. Assume $\mathcal{N}$ is loop-free. Let $\mathsf{bd}\,\mathcal{Z}$ denote the boundary of the zonotope $\mathcal{Z} := \pi([-1,1]^E)$. Then π restricts to a homeomorphism $\pi_{\mathcal{M}} : \|\Delta_{\mathcal{M}}\| \to \mathsf{bd}\,\mathcal{Z}$ if and only if $\mathcal{M} \rightsquigarrow \mathcal{N}$ is rank-preserving on flats.*

Proof. First, suppose $\pi_{\mathcal{M}} : \Delta_{\mathcal{M}} \to \mathsf{bd}\,\mathcal{Z}$ is a PL homeomorphism. To show there is a weak map $\mathcal{M} \rightsquigarrow \mathcal{N}$, it suffices to show that every tope of $\mathcal{N}$ is a tope of $\mathcal{M}$. First, note that each vertex v of $\mathcal{Z}$ has the property that $\pi^{-1}(v) = \{\mathbf{e}_T\}$ for some sign vector $T \in \mathbf{2}^E$. Indeed, $\pi^{-1}(v)$ is a face of $[-1,1]^E$, and if it were not a vertex of $[-1,1]^E$ then there would be some sign vector $T \in \mathbf{2}^E$, some $f \in E$, and some edge $[\mathbf{e}_T, \mathbf{e}_T - 2T(f)\mathbf{e}_f] \subseteq \pi^{-1}(v)$. But this means

$$0 = \pi(\mathbf{e}_T) - \pi(\mathbf{e}_T - 2T(f)\mathbf{e}_f) = 2T(f)\pi(\mathbf{e}_f) = 2T(f)v_f,$$

and so $v_f = 0$, which contradicts the assumption that $\mathcal{N}$ is loop free. Now, each vertex v_T of $\mathcal{Z}$ corresponds to some tope T of $\mathcal{N}$ in the sense that $v_T = \pi(\mathbf{e}_T)$. Hence, $\pi^{-1}(v_T) = \mathbf{e}_T$. Now, because $\pi_{\mathcal{M}}$ is surjective by assumption, there exists some $\alpha \in \Delta_{\mathcal{M}}$ such that $\pi(\alpha) = v_T$. Hence, $\alpha \in \pi^{-1}(v_T) = \{\mathbf{e}_T\}$. It follows that T is a tope of $\mathcal{M}$. The fact that $\mathcal{M} \rightsquigarrow \mathcal{N}$ is rank-preserving on flats follows from the assumption that $\pi_{\mathcal{M}}$ is a PL homeomorphism, and therefore, for each maximal flag of covectors $\mathscr{F} : \mathbf{0} = X_0 < \cdots < X_r$, we have

$$\pi_{\mathcal{M}}(\mathsf{conv}(\mathbf{e}_{X_1}, \ldots, \mathbf{e}_{X_r})) = \mathsf{conv}(v_{X_1}, \ldots, v_{X_r})$$

is an r-simplex. In particular, $v_{X_1}, \ldots, v_{X_r}$ are affinely independent, and hence linearly independent as they all lie on some facet of the full-dimensional 0-symmetric polytope

$\mathscr{Z}$. Thus Lemma 3.5.3 applies.

We now establish the other direction of the theorem. We proceed by induction on the rank r of $\mathcal{M}$. If $r = 0$, there is nothing to prove. Therefore, assume $r \geq 1$. Assume that there is a weak map $\mathcal{M} \rightsquigarrow \mathcal{N}$ that is rank-preserving on flats. We wish to show that $\pi_{\mathcal{M}} : \Delta_{\mathcal{M}} \to \mathsf{bd}\,\mathscr{Z}$ is a PL homeomorphism. To do this, it suffices to establish the three hypotheses of Theorem 3.4.8. We do this below:

1. Since $r \geq 1$, there is at least one cocircuit X of $\mathcal{M}$. Let $F = E \setminus X$ be the flat complementary to X. Then by Proposition 3.5.1 there is a weak map $\mathcal{M}|F \rightsquigarrow \mathcal{N}|F$ that is rank preserving on flats. Now, $\mathcal{N}|F$ is realized by the restriction of π to the subspace $\mathbf{R}^F$ of $\mathbf{R}^E$ spanned by $\{\mathbf{e}_f : f \in F\}$. Hence the linear map $\pi : \mathbf{R}^E \to \mathbf{R}^r$ restricts to a map of affine spaces $\pi : (\mathbf{e}_X + \mathbf{R}^F) \to \mathbf{R}^r$, which, in turn, restricts to a PL map $\pi_X : (\mathbf{e}_X + \Delta_{\mathcal{M}|F}) \to \mathsf{bd}\,\mathscr{Z}_X$ where

$$\mathscr{Z}_X = v_X + \sum_{f \in F} [-v_f, v_f]$$

is a translate of the zonotope of $\mathcal{N}|F$ by $v_X = \pi(\mathbf{e}_X)$. By induction, then, this map is a PL homeomorphism, which establishes (1).

2. This condition is immediate from Lemma 3.5.3.

3. Fix a regular point $y \in \mathsf{bd}\,\mathscr{Z}$, and $x \in \pi_{\mathcal{M}}^{-1}(y)$. Let

$$\mathscr{F} : X_1 < X_2 < \cdots < X_r$$
$$\mathscr{G} : Y_1 < Y_2 < \cdots < Y_r$$

denote the unique flags of covectors of $\mathcal{M}$ and $\mathcal{N}$, respectively, such that $x \in$

$\mathsf{im}\sigma$, $y \in \mathsf{im}\tau$, where

$$\sigma = \chi_{\mathcal{M}}(\mathscr{F})\sigma_{\mathscr{F}} \in H_k(\Delta_{\mathcal{M}})$$
$$\tau = \chi_{\mathcal{N}}(\mathscr{G})(\pi \circ \sigma_{\mathscr{G}}) \in H_k(\mathsf{bd}\,\mathcal{Z}).$$

Then the Jacobian determinant of $\tau^{-1}\pi_{\mathcal{M}}\sigma$ at $\sigma^{-1}(x)$ is equal to

$$\chi_{\mathcal{M}}(\mathscr{F})\det(v_{X_1},\ldots,v_{X_r})\left(\chi_{\mathcal{N}}(\mathscr{G})\det(v_{Y_1},\ldots,v_{Y_r})\right)^{-1},$$

and this is positive since

$$\chi_{\mathcal{N}}(\mathscr{G})\det(v_{Y_1},\ldots,v_{Y_r}) = |\det(v_{Y_1},\ldots,v_{Y_r})| > 0$$

and by Lemmas 3.5.2 and 3.5.3,

$$\chi_{\mathcal{M}}(\mathscr{F})\det(v_{X_1},\ldots,v_{X_r}) = \sum_S \chi_{\mathcal{N}}(S)^2\,|\det(v_{b_1},\ldots,v_{b_r})| > 0.$$

□

Remark 3.5.4. One half of the Bohne-Dress theorem can be deduced from this theorem as follows: Given a zonotope $\mathcal{Z} \subset \mathbf{R}^E$, let $\mathcal{N}$ denote the oriented matroid of $\mathcal{Z} \times [-1,1] \subset \mathbf{R}^E \times \mathbf{R}^e$; that is, $\mathcal{N}$ is the oriented matroid of $\mathcal{Z}$, plus a coloop e. Let $\mathcal{M}$ be an oriented matroid such that $\mathcal{M}/e = \mathcal{N}/e$. Then the identity map on E induces a weak map $\mathcal{M} \rightsquigarrow \mathcal{N}$ that is rank-preserving on flats. Applying Theorem 3.2.3 to $\mathcal{M}$ and $\mathcal{N}$, we obtain a triangulation of the boundary of the prism $\mathcal{Z} \times [-1,1]$ that realizes $\Delta_{\mathcal{M}}$. If we look in particular at one of the two big facets of $\mathcal{Z} \times [-1,1]$, we see a triangulation of $\mathcal{Z}$ which coincides with the canonical barycentric subdivision of a zonotopal tiling that represents $\mathcal{M}$.

Remark 3.5.5. The following sketch of an argument demonstrates that Theorem 3.2.3

can also be deduced from the Bohne-Dress theorem: Start with a realizable oriented matroid $\mathcal{N}$ and a weak map $\mathcal{M} \rightsquigarrow \mathcal{N}$ that is rank-preserving on flats. Let $\mathcal{Z}$ be a zonotope representing $\mathcal{N}$. Let $\tilde{\mathcal{N}} = \mathcal{N} + e$ be a realizable free extension of $\mathcal{N}$ and let $\tilde{\mathcal{M}}$ be an oriented matroid $\mathcal{M}$ such that there is a weak map $\tilde{\mathcal{M}} \rightsquigarrow \tilde{\mathcal{N}}$ that is rank-preserving on flats. Now, $\tilde{\mathcal{M}}$ is a single element lifting of $\tilde{\mathcal{N}}/e$, and therefore, by the Bohne-Dress theorem, there is a zonotopal tiling of the zonotope $\tilde{\mathcal{Z}}$ corresponding to $\tilde{\mathcal{N}}/e$ that represents $\tilde{\mathcal{M}}$. Taking the canonical barycentric subdivision of this zonotopal tiling, we obtain exactly one-half of a geometric realization of $\Delta_{\mathcal{M}}$. We then take two copies of this triangulated complex, multiply one of them by -1, and then lift both of them onto the boundary of $\mathcal{Z}$ so that they intersect on their common boundary. The result is a homeomorphic image of $\Delta_{\mathcal{M}}$ that lives on the boundary of $\mathcal{Z}$, whose support can be shown to be exactly $\pi(\Delta_{\mathcal{M}})$. The main challenge of this argument is to show that there is such a $\tilde{\mathcal{M}}$ that works; but it turns out that by simply taking the localization of $\tilde{\mathcal{N}}$ with respect to $\mathcal{N} + e$, and composing it with the localization of a generic lexicographic extension of $\mathcal{M}$, yields an oriented matroid with the desired properties (are you sure?). We leave the details to the interested reader.

Details

In this section we prove the following lemma, and then show how to derive Lemma 3.4.8 from it. Throughout this section, assume $k \geq 1$.

Lemma 3.6.1. *Let $\mathcal{T}$ be a triangulation of S^k, and let $f : S^k \to S^k$ be a map such that:*

1. *The degree of f is $+1$ or -1.*
2. *For each $\sigma \in \mathcal{T}$, the restriction $f|\sigma$ is a homeomorphism onto its image.*

3. There exists some sign $s \in \{-1, 1\}$ such that for each maximal cell $\sigma \in \mathcal{T}$, the local degree of f at each point in the interior of σ has sign s.

Then $f : S^k \to S^k$ is a homeomorphism.

Preliminaries

The proof of this lemma relies on the following result due to Brouwer:

Theorem 3.6.2 (Invariance of Domain [41, Theorem 2B.3])**.** *Let $f : S^k \to S^k$ be a map, and let $U \subseteq S^k$ be open such that the restriction $f : U \to f(U)$ is injective. Then $f(U)$ is open in S^k.*

We also need the notion of local degree:

Definition 3.6.3 ([42, Definition 5.1])**.** Let $V \subset S^k$ be an open set, and let $y \in S^k$. Let $f : V \to S^k$ be such that $f^{-1}(y)$ is compact. Then define the *local degree of f at y*, denoted $\deg_y(f)$ to be the integer d such that the composite

$$H_k(S^k) \longrightarrow H_k(S^k|f^{-1}(y)) \longrightarrow H_k(V|f^{-1}(y)) \xrightarrow{f_*} H_k(S^k|y) \longrightarrow H_k(S^k)$$

is given by $x \mapsto d \cdot x$. Here the first map is the projection map, the second is an excision map, and last is the inverse of the projection map. The notation $H_k(X|A)$ denotes the relative homology group $H_k(X, X - A)$.

Proposition 3.6.4 ([42, Proposition 5.8])**.** *Let $V \subseteq S^k$ be an open set and let $f : V \to S^k$. Suppose we can write V as a union*

$$V = V_1 \cup V_2 \cup \cdots \cup V_m$$

where each V_i is open. Let $f_i : V_i \to S^k$ denote the restriction of f to V_i for each $i = 1, \ldots, m$. Suppose $y \in S^k$ has the property that $f_i^{-1}(y)$ is disjoint from $f_j^{-1}(y)$ for *all $i \neq j$. Then*

$$\deg_y(f) = \sum_{i=1}^{m} \deg_y(f_i).$$

Definition 3.6.5 ([42, Definition 5.11])**.** Let $V \subseteq S^k$ be an open set and let $f : V \to S^k$. Let $W \subseteq S^k$. Then f is *proper over W* if $f^{-1}(L)$ is compact for every compact $L \subset W$.

Proposition 3.6.6 ([42, Proposition 5.12])**.** *Let $V \subseteq S^k$ be an open set and let $f : V \to S^k$. Let W be a connected open set such that f is proper over W. Then the function $W \to \mathbf{Z}$ given by $y \mapsto \deg_y(f)$ is constant.*

<u>Proof of Lemma 3.6.1</u>

In the following three propositions, we assume $\mathcal{T}$ and $f : S^k \to S^k$ are as in the statement of the lemma.

Proposition 3.6.7. *Let $y \in S^k$. Let $f^{-1}(y) = \{x_1, \ldots, x_s\}$. Let U_i be a neighbourhood of x_i for $i = 1, 2, \ldots, s$, so that the U_i's are pairwise disjoint. Then there exists some $\varepsilon > 0$ such that*

$$f^{-1}(B(y, \varepsilon)) \subseteq U_1 \cup U_2 \cup \cdots \cup U_s.$$

Here $B(y, \varepsilon)$ denotes the open cap in S^k centered at y of radius ε.

Proof. Let $V_n = f^{-1}(B(y, \frac{1}{n}))$. Note that $V_1 \supset V_2 \supset V_3 \supset \cdots$ We have that

$$f^{-1}(y) = \bigcap_{n \geq 1} V_n$$

since a point $x \in S^k$ lies in this intersection if and only if $\|f(x) - y\| < 1/n$ for all $n \geq 1$; that is, $f(x) = y$. Now Let $U = U_1 \cup \cdots \cup U_s$ and let $K = S^k \smallsetminus U$. We have an open cover of K as follows: each $x \in K$ is covered by $S^k \smallsetminus V_n$ where $n = n(x)$ is the smallest integer for which $x \notin V_n$. Since K is a closed subset of S^k, and therefore compact, this open cover has a finite subcover; in particular, there exists some N such that $S^k \smallsetminus U = K \subseteq S^k \smallsetminus V_N$, and therefore we conclude $V_N \subseteq U$. $\square$

Proposition 3.6.8. *Define the set*

$$X := f\left(\bigcup_{\sigma\in\mathcal{T}} \mathsf{bd}\,\sigma\right) \subseteq S^k.$$

Then $S^k \smallsetminus X$ is open and dense in S^k.

Proof. We start by showing that $S^k \smallsetminus f(\mathsf{bd}\,\sigma)$ is open and dense in S^k for all $\sigma \in \mathcal{T}$. That $S^k \smallsetminus f(\mathsf{bd}\,\sigma)$ is open is clear from the fact that $f : S^k \to S^k$ maps closed sets to closed sets, and $\mathsf{bd}\,\sigma$ is closed. To see that $S^k \smallsetminus f(\mathsf{bd}\,\sigma)$ is dense, Let $y \in f(\mathsf{bd}\,\sigma)$. There exists a unique $x \in \mathsf{bd}\,\sigma$ such that $f(x) = y$. Since σ is homeomorphic to a closed ball, we can find a sequence $x_1, x_2, \ldots$ in σ° which converges to x. By continuity of f, we have $f(x_n) \to f(x) = y$, and since $f|\sigma$ is a homeomorphic to $f(\sigma)$, we must have that each $f(x_n) \in f(\sigma^\circ)$ which is disjoint from $f(\mathsf{bd}\,\sigma)$. We conclude that y lies in the closure of $S^k \smallsetminus f(\mathsf{bd}\,\sigma)$. Since $|\mathcal{T}|$ is finite, the conclusion of the proposition follows from the fact that a finite intersection of open dense sets in S^k is open dense. $\square$

Proposition 3.6.9. *Let W be open in S^k, and let $V = f^{-1}(W)$. Suppose V can be written as the disjoint union $V = V_1 \cup \cdots \cup V_s$ where each V_i is open. Then the restriction $f_i : V_i \to S^k$ is proper over W for each $i = 1, 2, \ldots, s$.*

Proof. Suppose $i = 1, 2, \ldots, s$, and let $L \subseteq W$ be a compact set. We show that each $f_i^{-1}(L)$ is closed in S^k, and hence compact, by showing that $f_i^{-1}(L)$ contains all its limit points.

Let $\{x_n\}$ be a convergent sequence in $f_i^{-1}(L) = f^{-1}(L) \cap V_i$ which converges to some $x \in S^k$. Since $f_i^{-1}(L) \subseteq f^{-1}(L)$ which is closed, we have $x \in f^{-1}(L)$. Hence it remains to show $x \in V_i$. We may write

$$f^{-1}(L) = \bigcup_{i=1}^{s} f^{-1}(L) \cap V_i = \bigcup_{i=1}^{s} f_i^{-1}(L),$$

which means in particular that $x \in f_j^{-1}(L)$ for some $j = 1, 2, \ldots, s$. If $i \neq j$, then there exists some $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq V_j$, and hence each x_n in the sequence $x_1, x_2, \ldots$ has distance at least ε from x. This contradicts the fact that $x_n \to x$. We conclude that $i = j$. $\square$

We are now ready to prove the lemma.

Proof of Lemma. Let $y \in S^k$. Then $f^{-1}(y) = \{x_1, \ldots, x_t\}$. Let U_i be a neighbourhood of x_i for all $i = 1, 2, \ldots, t$, so that the U_i's are pairwise disjoint, and let $U = U_1 \cup U_2 \cup \cdots \cup U_t$. Let $W = B(y, \varepsilon) \subseteq S^k$, where $\varepsilon > 0$ is chosen small enough so that $V := f^{-1}(W)$ is a neighbourhood of $f^{-1}(y)$ contained in U. Such an ε exists by Proposition . Let $V_i = V \cap U_i$ for all $i = 1, 2, \ldots, t$. Then the V_i's are also pairwise disjoint, and each V_i is an open neighbourhood of x_i.

Choose some $\sigma \in \mathcal{T}$ containing x_i. Then $V_i \cap \sigma^\circ$ is open in S^k and nonempty, since it is possible to approach x_i from within σ°. Moreover, since the map $f|_\sigma : \sigma \to f(\sigma)$ is a homeomorphism, any restriction of this map is also a homeomorphism. From this and Theorem we conclude that $f(V_i \cap \sigma^\circ)$ is open in S^k.

Now, let $X = f(\cup_{\tau \in \mathcal{T}} \operatorname{\mathsf{bd}} \tau)$, so that $S^k \setminus X$ is open and dense in S^k by Proposition . Then $f(V_i \cap \sigma^\circ)$ intersects $S^k \setminus X$ at a point z_i. Write $f^{-1}(z_i) \cap V_i = \{w_{i1}, w_{i2} \ldots, w_{i,\ell(i)}\}$. Note that since $z_i \in f(V_i)$, we have $\ell(i) \geq 1$. Let W_{ij} be an open neighbourhood of w_{ij} in V_i, and let $f_{ij} : W_{ij} \to S^k$ denote the restriction of f

to W_{ij} for all $j = 1, 2, \ldots, \ell(i)$. Let $s = \deg(f)$. We have

$$\begin{aligned} 1 = s \cdot \deg(f) &= \sum_{i=1}^{t} s \cdot \deg_y(f_i) \\ &= \sum_{i=1}^{t} s \cdot \deg_{z_i}(f_i) \\ &= \sum_{i=1}^{t} \sum_{j=1}^{\ell(i)} s \cdot \deg_{z_i}(f_{ij}) \geq \ell(1) + \ell(2) + \cdots + \ell(t) \geq t. \end{aligned}$$

The first equality holds by assumption (1), the second equality holds by Proposition , the third equality holds by Propositions and , and the fourth equality holds again by Proposition . The second-to-last inequality holds by assumption (3), and the last inequality holds since each $\ell(i) \geq 1$. Since t is a positive number, we must therefore have $t = 1$. Since y was arbitrarily chosen, we conclude f must be injective, and therefore a homeomorphism. $\square$

Proof of Lemma 3.4.8

Definition 3.6.10. Let $U \subset \Delta_k$ be open. A map $\varphi : U \to \Delta_k$ is is *orientation preserving at* $x \in U$, if the diagram below commutes:

$$\begin{tikzcd} H_k(\Delta_k|x) \arrow[r] \arrow[dr] & H_k(U|x) \arrow[d, "\varphi_*"] \\ & H_k(\Delta_k|\varphi(x)) \end{tikzcd}$$

Here the top map is the excision isomorphism, and the diagonal sends the class of the identity map $\mathbf{1} : \Delta_k \to \Delta_k$ in $H_k(\Delta_k|x)$ to the class of the identity map in $H_k(\Delta_k|\varphi(x))$

Lemma 3.6.11. *Let $\mathcal{T}_1, \mathcal{T}_2$ be two triangulations of S^k, and suppose that $H_k(S^k)$ is generated by*

$$\alpha = \sum_{\sigma \in \mathcal{T}_1} \sigma = \sum_{\tau \in \mathcal{T}_2} \tau.$$

Suppose that $f : S^k \to S^k$ has the following property: For each $y \in S^k \smallsetminus \mathsf{bd}\,\mathcal{T}_2$, and for each $x \in f^{-1}(y) \smallsetminus \mathsf{bd}\,\mathcal{T}_1$, the map $\varphi : U_0 \to \Delta_k$ is orientation preserving, where:

- *σ, τ are the unique maps of $\mathcal{T}_1, \mathcal{T}_2$, respectively, such that $x \in \mathsf{im}\sigma$ and $y \in \mathsf{im}\tau$,*
- *$U \subset \mathsf{im}\sigma$ is a neighbourhood of x such that $U \cap f^{-1}(y) = \{x\}$,*
- *$U_0 = \sigma^{-1}(U)$,*
- *The map φ is the composite $\tau^{-1} \circ f \circ (\sigma|_{U_0})$.*

Then the local degree of the restiction $f : U \to S^k$ at x is equal to 1.

Proof. Let $y_0 = \tau^{-1}(y)$, and consider the following diagram:

$$\begin{tikzcd}
H_k(S^k) \arrow[r] & H_k(S^k|x) \arrow[r] & H_k(U|x) \arrow[r, "f_*"] & H_k(S^k|y) \arrow[r] & H_k(S^k) \\
& H_k(\Delta_k|x_0) \arrow[u, "\sigma_*"] \arrow[r] \arrow[rr, bend right, "\mathbf{1}\mapsto\mathbf{1}"'] & H_k(U_0|x_0) \arrow[u, "\sigma_*"] \arrow[r, "\varphi_*"] & H_k(\Delta_k|y_0) \arrow[u, "\tau_*"'] &
\end{tikzcd}$$

Let $\gamma \in H_k(U_0|x_0)$ be the cycle that is the image of $[\mathbf{1}]$ under the bottom-left excision map. Then, since φ is orientation preserving, we have $[\mathbf{1}] = [\varphi \circ \gamma]$. Commutativity of the right square implies $[f \circ \sigma \circ \gamma] = [\tau \circ \varphi \circ \gamma] = [\tau]$. Hence, under the maps of the top row, α is mapped as follows:

$$\alpha \longmapsto [\sigma] \longmapsto [\sigma \circ \gamma] \longmapsto [f \circ \sigma \circ \gamma] = [\tau \circ \varphi \circ \gamma] = [\tau] \longmapsto \alpha,$$

and hence the local degree of φ at x equals 1. $\square$

In the following lemma, let A denote the affine span of Δ_k, which is the hyperplane $\sum x_i = 1$ inside $\mathbf{R}^{k+1}$. Given an affine map $B : A \to A$, there exists a unique linear map $\tilde{B} : \mathbf{R}^{k+1} \to \mathbf{R}^{k+1}$ such that the restriction of $\tilde{B}$ to A is B.

Lemma 3.6.12. *Let $U \subset \Delta_k$ be open, and let $B : A \to A$ be an affine map such that $B(U) \subset \Delta_k^\circ$. Let $\varphi : U \to \Delta_k^\circ$ denote the restriction of B to U. If $\det \tilde{B} > 0$, then φ is orientation preserving at each $x \in U$.*

Proof. Let $x \in U$, and let $y = \varphi(x)$. Note that since $\det \tilde{B} > 0$, we must have $\tilde{B}(A) = A$. Since $\mathrm{SL}(\mathbf{R}^{k+1})$ is path-connected, we can find a path $\tilde{B}_t$, $t \in [0,1]$, of positive-determinant matrices such that $\tilde{B}_t(A) = A$ and $\tilde{B}_0 = I$ and $\tilde{B}_1 = \tilde{B}$. From this we get a homotopy of maps $\varphi_t : (A, A - x) \to (A, A - y)$, $t \in [0,1]$ given by

$$\varphi_t(v) = \tilde{B}_t(v - x) + y$$

so that φ_0 is the translation mapping $T : A \to A$ given by $T(v) := v + y - x$, and $\varphi_1 = B$. From this we conclude $T^* = B^* : H_k(A|x) \to H_k(A|y)$, by Proposition 2.19 in Hatcher.

Now let S be the line segment joining x and y, and consider the homotopy $T_t : (\Delta_k, \Delta_k - S) \to (A, A - y)$, $t \in [0,1]$ given by

$$T_t(v) = v + t(y - x).$$

Then T_0 is the inclusion map $i_A : \Delta_k \to A$, while $T_1 = T|_{\Delta_k}$. It follows that $(i_A)_* = (T|_{\Delta_k})_* : H_k(\Delta_k|S) \to H_k(A|y)$, again, by Proposition 2.19. Now consider the following diagram:

$$\begin{tikzcd}
H_k(\Delta_k|y) & H_k(\Delta_k|S) \arrow[l] \arrow[r] \arrow[d, "(i_A)_*=(T|_{\Delta_k})_*"] & H_k(\Delta_k|x) \arrow[d] \arrow[dr] & \\
 & H_k(A|y) \arrow[ul] & H_k(A|x) \arrow[l, "T_*=B_*"'] & H_k(U|x) \arrow[l]
\end{tikzcd}$$

We show that φ is orientation preserving at each $x \in U$. Let $[\gamma] \in H_k(U|x)$ denote the image of $[\mathbf{1}]$ under the rightmost diagonal map. We have $[\gamma]$ is sent to

$[i_A \circ \gamma] = [i_A]$ under the bottom right map, by commutativity of the right triangle. This in turn is mapped to

$$[i_A \circ \varphi \circ \gamma] = [B \circ i_A \circ \gamma] = [B \circ i_A] = [T \circ i_A].$$

On the other hand, we also have

$$[T \circ i_A] = [(T|_{\Delta_k}) \circ \mathbf{1}] = [i_A \circ \mathbf{1}] = [i_A].$$

So we conclude that $[i_A] = [i_A \circ \varphi \circ \gamma] \in H(A|y)$. Since the leftmost diagonal map is an isomorphism, this implies that $[\mathbf{1}] = [\varphi \circ \gamma] \in H(\Delta_k|x)$. $\square$

Finally, we give a sufficient condition for a map $f : S^k \to S^k$ to have degree $+1$ or -1.

Lemma 3.6.13. *Let $S_0 \subset S^k$ be homeomorphic to S^{k-1}, so that there is a homeomorphism $S^k \to S^k$ which maps S_0 to the equator of S^k. Suppose $f : S^k \to S^k$ is a surjective map for which $f(S_0) = X \cap Y$, where X, Y are the closures in S^k of the two components of $S^k \setminus S_0$. Further suppose $f(S_0)$ is homeomorphic to S^{k-1}. Then the degee of f is $+1$ or -1.*

Proof. We can slightly large the upper and lower hemispheres of S^k to obtain open $U, V \subset S^k$ for which $S^k = U \cup V$, so that there is a deformation retraction of U onto the closed upper hemisphere X of S^k, and similarly there is a deformation retraction of V onto the closed lower hemisphere V of S^k. There is also a deformation retraction of $U \cap V$ onto $X \cap Y = S_0$. Now, consider the diagram

$$\begin{array}{ccccccc}
\tilde{H}_k(X) \oplus \tilde{H}_k(Y) & \longrightarrow & \tilde{H}_k(S^k) & \longrightarrow & \tilde{H}_{k-1}(S_0) & \longrightarrow & \tilde{H}_{k-1}(X) \oplus \tilde{H}_{k-1}(Y) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\tilde{H}_k(f(X)) \oplus \tilde{H}_k(f(Y)) & \longrightarrow & \tilde{H}_k(S^k) & \longrightarrow & \tilde{H}_{k-1}(f(S_0)) & \longrightarrow & \tilde{H}_{k-1}(f(X)) \oplus \tilde{H}_{k-1}(f(Y))
\end{array}$$

The top and bottom rows are portions of Mayer-Vietoris exact sequences. By naturality, this diagram commutes. Now all three of $\tilde{H}_k(S^k)$, $\tilde{H}_{k-1}(S_0)$, and $\tilde{H}_{k-1}(f(S_0))$ are isomorphic to $\mathbf{Z}$, and since X, Y are contractible, by exactness the top middle map is an isomorphism. It follows that the map

$$\tilde{H}_k(S^k) \longrightarrow \tilde{H}_{k-1}(S_0) \xrightarrow{f_*} \tilde{H}_{k-1}(f(S_0))$$

is an isomorphism. By commutativity of the middle square, we conclude that $f_* : \tilde{H}_k(S^k) \to \tilde{H}_k(S^k)$ is an isomorphism, and hence the degree of f is $+1$ or -1. $\square$

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