1. Computability, Complexity and Algorithms

Given an undirected graph $G = (V, E)$ with n vertices, two vertices $s, t \in V$ and an integer N, the #paths problem is to determine whether there exist at least N distinct s-t simple paths in G (note we say distinct, not disjoint). Use the following steps to show the problem is NP-complete by a reduction from the Hamitonian cycle problem.

1. Show that the number of simple cycles through a given edge of a given graph G can be counted by a reduction to the #paths problem.
2. Given a simple undirected graph $H = (U, F)$, let H' be obtained by subdividing each edge into ℓ edges and creating k parallel copies of each edge. Take a single Hamiltonian cycle in H. How many distinct Hamiltonian cycles in H' does it map to?
3. Suppose H is Hamiltonian and H' is constructed with $k = n, \ell = n+c$. Show that the total number of non-Hamiltonian simple cycles in H' is smaller than the number of Hamiltonian cycles in H' by a factor of n^c.
4. Show that the problem of deciding whether a given graph has a Hamiltonian cycle can be reduced to the #paths problem in polynomial time.

2. Theory of Linear Inequalities

Let x^* be a fractional extreme point of a rational polytopte $P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$. Prove that there exists a Chvátal-Gomory cut for P that separates x^*.

3. Graph Theory

Let G be a 2-connected plane graph and let $V(G), E(G), F(G)$ denote its set of vertices, set of edges, set of faces, receptively. Let $\sigma : V(G) \cup F(G) \to \mathbb{Z}$ such that $\sigma(x) = d(x) - 4$ for all $x \in V(G) \cup F(G)$, where $d(x)$ is the number of edges incident with x. Show that

(1) $\sum_{x \in V(G) \cup F(G)} \sigma(x) = -8$.

(2) If $\delta(G) \geq 5$ then G contains K_4^- (obtained from K_4 by removing an edge) as a subgraph.

4. Algebra

(a) Let R be an integral domain containing a field k as a subring. Suppose that R is a finite dimensional vector space over k under the ring multiplication. Show that R is a field.

(b) Show that the conclusion does not hold without the assumption of being finite dimensional. That is, give an example of a ring R containing a field k as a subring, such that R is an integral domain but is not a field.

(c) Show that the conclusion does not hold without the assumption of being an integral domain. That is, give an example of a ring R containing a field k as a subring, such that R is finite dimensional over k but is not a field.

4. Linear Algebra

Let A, B be $n \times n$ matrices. Show that $\sigma(AB) = \sigma(BA)$. (Recall the spectrum of A, $\sigma(A) = \{\lambda : A - \lambda I \text{ is not invertible}\}$.)

5. Analysis of Algorithms

Part a: Let $f(x)$ be a real-valued function. You are given a randomized algorithm B that, given input x and parameter $\epsilon > 0$, outputs $B(x)$ which approximates $f(x)$ as follows:

$$\forall x, \quad \mathbf{Pr}\left[(1-\varepsilon)f(x) \le B(x) \le (1+\varepsilon)f(x)\right] \ge 3/4. \tag{1}$$

Give an algorithm C that, given input x and parameters $\epsilon, \delta > 0$, outputs $C(x)$ satisfying:

$$\forall x, \quad \mathbf{Pr}\left[(1-\varepsilon)f(x) \le C(x) \le (1+\varepsilon)f(x)\right] \ge 1-\delta. \tag{2}$$

Achieve the best dependence on δ in $O(\cdot)$ notation (i.e., ignore constant factors).
Part b: Explain if your approach in (a) still works if instead of (**??**) the probability of success is weakened so that:

$$\forall x, \quad \mathbf{Pr}\left[(1-\varepsilon)f(x) \le B(x) \le (1+\varepsilon)f(x)\right] \ge 1/4. \tag{3}$$

Part c: Suppose that we have n polygons $P_1, \dots, P_n$, all lying inside $[0,1] \times [0,1]$ which is the square with side length 1 on the Euclidean plane. Every polygon has area at least $\alpha > 0$. You are not given the polygons explicitly but instead for each polygon P_i we have access to a membership oracle: given a point $x \in \mathbf{R}^2$, the oracle returns YES if $x \in P_i$ and NO if $x \notin P_i$.

Give a randomized algorithm that approximately estimates the area of the *union* of these polygons. Given $0 < \varepsilon < 1$, the output Y of your algorithm should satisfy

$$\mathbf{Pr}\left[(1-\varepsilon)|P| \le Y \le (1+\varepsilon)|P|\right] \ge \frac{3}{4}$$

where $|P|$ denotes the area of $P = P_1 \cup \cdots \cup P_n$.
(You may assume that sampling a real number uniformly at random from $[0,1]$ takes constant time, and that each oracle call takes constant time.)
The running time of your algorithm should be polynomial in $n, 1/\varepsilon$, and $1/\alpha$.

6. Combinatorial Optimization

Recall that a graph G is *factor-critical* if for all $v \in V(G)$, $G - v$ has a perfect matching. An *open odd ear decomposition* of G is a sequence $H_0, H_1, \ldots, H_k$ of subgraphs of G such that, letting $G_j = \bigcup_{i=0}^{j} H_i$ for $j = 0, 1, \ldots, k$, we have

(a) G_0 is an odd cycle,

(b) for $i = 1, 2, \ldots, k$ the graph H_i is an odd length (i.e., odd number of edges) path with both (distinct) end vertices in $V(G_{i-1})$ and no internal vertex or edge in $V(G_{i-1})$, and

(c) $G = G_k$.

Show that if a 2-connected graph G is factor-critical, then it admits an open odd ear decomposition.

7. Probabilistic methods

Prove that there is some constant $c > 0$ so that for every integer $k \geq 1$, given a graph and a set of k acceptable colors for each vertex such that every color is acceptable for at most ck neighbors of each vertex, there is always a *proper coloring* where every vertex is assigned one of its acceptable colors. (Recall as usual that a proper coloring requires that the endpoints of every edge get different colors.)