1. Computability, Complexity and Algorithms

Bottleneck edges in a flow network:

Consider a flow network on a directed graph $G = (V, E)$ with capacities $c_e > 0$ for $e \in E$. An edge $e \in E$ is called a *bottleneck edge* if increasing the capacity c_e increases the size of the maximum flow.

Given a flow network $G = (V, E)$ and a maximum flow f^*, give an algorithm to identify *all* bottleneck edges. Do as fast in $O(\cdot)$ as possible. Justify correctness of your algorithm. You can assume basic operations (comparison, addition, subtraction, multiplication, and division) on two numbers take constant time.

2. Analysis of Algorithms

All-pairs shortest paths (APSP) and Min-Sum Products. Suppose W is the adjacency matrix for G a simple undirected graph with no self-loops and no negative edge weights, and W^* is the reachability matrix ($w^*_{ij} = 1$ if there exists a path from i to j).

- Suppose operations are boolean (addition is OR, multiplication is AND). Suppose

  $$W = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

  Then show that

  $$W^* = \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} (A \vee BD^*C)^* & EBD^* \\ D^*CE & D^* \vee GBD^* \end{bmatrix}$$

  Observe that F, G use E in their definition, etc., so the calculations have to be done in the correct order. *Hint:* Consider G as partitioned into two subcomponents $V = V_1 \uplus V_2$.

- Now suppose W_{ij} is the weight of the edge (i, j). Moreover, now assume that matrix products are min-sum products (that is, addition is replaced by min and product by sum), and $A \vee B$ is the element-wise minimum of matrices A and B. If W^*_{ij} now denotes the shortest-path distance from i to j, show that W^* is computed by the same relation as in the previous part. You may be brief, 2-3 sentences suffices if your previous answer was thorough.

- Using this idea, show that

  $$\text{APSP}(n) \leq 2\text{APSP}(n/2) + 6\text{MSP}(n/2) + O(n^2), \tag{1}$$

  where APSP(n) denotes the worst-case running time of computing APSP on an n-vertex input graph, and MSP(n) denotes the worst-case running time of computing the min-sum product of two $n \times n$ matrices. Assume that arithmetic operations can be carried out in constant time.

  In turn, show that $\text{APSP}(n) = \tilde{O}(\text{MSP}(n) + n^2)$. *Hint:* We know that MSP is superlinear, even superquadratic, in its runtime, simply since it needs to read its two input matrices.

3. Theory of Linear Inequalities

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\} \subseteq [0,1]^n$ be a polytope with 0/1 vertices. It is well known that the diameter of any 0/1 polytope is at most n. Here we consider a stronger notion of diameter where the sequence of vertices has to be non-decreasing in value with respect to a given objective $c \in \mathbb{Z}^n$: For any two vertices $x, y \in P$ with $cy = \max_{z \in P} cz$ find the shortest path of *adjacent* vertices $x_1, \ldots, x_l$ with $x = x_1$ and $y = x_l$ so that $cx = cx_1 \leq \cdots \leq cx_l = cy$. The *monotone diameter* for an objective c is the maximum length over all such vertex pairs.

Prove that the monotone diameter is at most $O(n \log C)$, where $C = \max_i |c_i|$ (6 points). Can you also show that in this case the monotone diameter is at most n irrespective of the objective c? (4 points)

4. Combinatorial Optimization

Let $\mathcal{M} = (U, \mathcal{I})$ be a matroid and $w : U \to \mathbb{R}$ be a weight function.

1. Given any two bases B and B', show that there exists a sequence of bases $B_0, B_1, \ldots, B_k$ with the following properties.
   (a) $B_0 = B$ and $B_k = B'$.
   (b) $B_i \subseteq B \cup B'$ for each $0 \leq i \leq k$.
   (c) $|B_i \Delta B_{i+1}| = 2$ for each $0 \leq i \leq k-1$.
2. Suppose B' is a maximum weight basis under weight function w. Show that we can additionally ensure that $w(B_{i+1}) \geq w(B_i)$ for each $0 \leq i \leq k-1$.

5. Graph Theory

Let G be a 2-connected graph and let $s \in V(G)$. Prove that G has two spanning trees T_1, T_2 such that for every vertex $v \in V(G)$ the two paths between v and s in T_1 and T_2 are internally disjoint.

6. Probabilistic methods

Suppose that we throw m balls into n bins independently and uniformly at random (initially all bins are empty, of course).

(A) Prove that $m^*(n) = n \log n$ is a threshold function for the property 'there exists an empty bin', i.e.,

$$\Pr(\text{there exists an empty bin}) \to \begin{cases} 1 & m \ll n \log n, \\ 0 & m \gg n \log n. \end{cases}$$

(B) Make an educated guess what the threshold function for the property 'there exists a bin with at most one ball' is. Prove the corresponding 0-statement (no proof of the corresponding 1-statement expected).

Hint: *Recall that* $1 - x = e^{-x+O(x^2)}$ *as* $x \to 0$.

7. Algebra

Suppose p and q are odd primes and $p < q$. Let G be a finite group of order p^3q. Prove that G has a normal Sylow subgroup.