6-CONNECTED GRAPHS ARE TWO-THREE LINKED

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No, emptiness is not nothingness. Emptiness is a type of existence. You must use this existential emptiness to fill yourself.

Liu Cixin, The Three-Body Problem

To my parents and my wife.

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SUMMARY

Let G be a graph and a_0, a_1, a_2, b_1, and b_2 be distinct vertices of G. Motivated by their work on Jørgensen's conjecture, Robertson and Seymour asked when does G contain disjoint connected subgraphs G_1, G_2, such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$ and $\{b_1, b_2\} \subseteq V(G_2)$. We prove that if G is 6-connected then such G_1, G_2 exist. Joint work with Robin Thomas and Xingxing Yu.

CHAPTER 1

INTRODUCTION AND BACKGROUND

The Four Color Theorem [1, 2, 3] asserts that every loopless planar graph admits a vertex 4-colouring. The related problem was first put forward by Francis Guthrie in 1852, who asked whether it was true that any planar map can be colored with four colors such that adjacent regions receive different colors. In 1976, Appel and Haken [1] claimed a proof of the Four Color Theorem with the help of a computer. However, some computer-free parts of their proof are complicated and tedious to verify. In 1997, Robertson, Sanders, Seymour, and Thomas [2, 3] gave a much simpler proof for the Four Color Theorem.

According to Kuratowski's theorem [4], a graph is planar if and only if it contains no K_5-subdivision or $K_{3,3}$-subdivision. Moreover, it is well known that any 3-connected nonplanar graph other than K_5 contains a $K_{3,3}$-subdivision. Hence, as an extension of the Four Color Theorem, it is natural to ask whether every graph without K_5-subdivision is also 4-colorable. More generally, Hajós [5] conjectured that for any positive integer k, every graph containing no K_{k+1}-subdivision is k-colorable. This conjecture is true for $k \leq 3$, but Catlin [5] found counterexamples to this conjecture for each $k \geq 6$. However, the cases for $k = 4$ and $k = 5$ are still open. Efforts have been made to resolve Hajós' conjecture for $k = 4$. Yu and Zickfeld [6] proved that a minimum counterexample to Hajós' conjecture when $k = 4$ must be 4-connected. Moreover, Sun and Yu [7] showed that if G is a minimum counterexample to Hajós' conjecture and S is a 4-cut in G then $G - S$ has exactly two components. In fact, if one can show a minimum counterexample to Hajós' conjecture for $k = 4$ is 5-connected, then Hajós' conjecture for $k = 4$ will immediately follow from the Kelmans-Seymour conjecture [8, 9]: Every 5-connected nonplanar graph contains K_5-subdivision. This Kelmans-Seymour conjecture was recently proved by He, Wang, and Yu [10, 11, 12, 13].

While Hajós' conjecture concerns the chromatic number of graphs without K_{k+1}-subdivision, Hadwiger [14], in 1943, conjectured a far-reaching generalization of the Four Color Theorem in terms of K_{k+1}-minor: For any positive integer k, if a graph contains no K_{k+1}-minor then it is k-colorable.

It is easy to prove that Hadwiger's conjecture holds for $k \leq 2$. Hadwiger [14] and Dirac [15] proved the case for $k = 3$. For $k = 4$, Hadwiger's conjecture is equivalent to the Four Color Theorem by the result of Wagner [16], which characterized graphs containing no K_5-minor and showed that Four Color Theorem implies that graphs containing no K_5-minor are 4-colorable. The case $k = 5$ can also be reduced to the Four Color Theorem, as shown by Robertson, Seymour, and Thomas [17]. However, this conjecture remains open for $k \geq 6$.

In fact, there are also many other interesting results related to Hadwiger's conjecture. Suppose Hadwiger's conjecture is false for some k, and let G be a minor minimal counterexample. Dirac [15] showed that G is 5-connected when $k \geq 5$, and Mader [18] showed that G is 6-connected when $k \geq 5$, and 7-connected when $k \geq 6$. Kawarabayashi and G. Yu [19] proved that G is $(2k/27)$-connected, improving upon an earlier bound in [20].

Let the *stability number* $\alpha(G)$ of a graph G denote the size of the largest stable set. Then every n-vertex graph G has chromatic number at least $\lceil n/\alpha(G) \rceil$, and should contain a clique minor of this size if Hadwiger's conjecture is true. In 1982, Duchet and Meyniel [21] proved that every n-vertex graph G has a K_k minor where $k \geq n/(2\alpha(G) - 1)$. Moreover, there has been a subsequent improvement by Fox [22]. And then Balogh and Kostochka [23] further improved the result, and showed that every n-vertex graph G has a K_k minor where $k \geq 0.51338n/\alpha(G)$. Later, in 2007, Kawarabayashi and Song [24] proved that every n-vertex graph G with $\alpha(G) \geq 3$ has a K_k minor where $k \geq n/(2\alpha(G) - 2)$.

For an n-vertex graph G with $\alpha(G) = 2$, the Duchet-Meyniel theorem implies that there is a K_k minor with $k \geq n/3$, which was strengthened by Böhme, Kostochka and Thomason [25] in 2011. They proved that every n-vertex graph with chromatic number t

has a K_k minor where $k \geq (4t - n)/3$.

A graph is *claw-free* if no vertex has three pairwise nonadjacent neighbours. So graphs with stability number two are claw-free. Fradkin [26] showed that every n-vertex connected claw-free graph G with $\alpha(G) \geq 3$ has a K_k minor where $k \geq n/\alpha(G)$. Furthermore, in 2010, Chudnovsky and Fradkin [27] proved that every claw-free graph G with no K_{k+1} minor is $\lfloor 3k/2 \rfloor$-colourable.

Since line graphs are claw-free, these results about claw-free graphs are related to a theorem of Reed and Seymour. They showed [28] that Hadwiger's conjecture is true for line graphs (of multigraphs).

We say that H is an *odd minor* of G if H can be obtained from a subgraph G' of G by contracting a set of edges that is a cut of G'. Clearly, a graph contains K_3 as an odd minor if and only if it is not 2-colourable. In 1979, Catlin [5] showed that if G has no K_4 odd minor then G is 3-colourable.

A *fully odd* K_4 in G is a subgraph of G which is obtained from K_4 by replacing each edge of K_4 by a path of odd length in such a way that the interiors of these six paths are disjoint. Then in 1998, Zang [29] proved (and, independently, Thomassen [30] proved in 2001) the conjecture of Toft [31] that if G contains no fully odd K_4 then G is 3-colourable.

Moreover, in 1995, Gerards and Seymour conjectured a strenthening of Hadwiger's conjecture (see [32]) that for every $k \geq 0$, if G has no K_{k+1} odd minor, then G is k-colourable, and it is known as true for $k \leq 3$.

In fact, one can find more interesting results and open problems about Hadwiger's conjecture and its variations from a survey [33], written by Seymour in 2016.

Now, we just go back and spend a bit more space on the $k = 5$ case of the Hadwiger conjecture. As we mentioned, Mader [18] proved that any minor minimal counterexample to the Hadwiger conjecture for $k = 5$ is 6-connected. Jørgensen [34] conjectured that every 6-connected graph contains a K_6-minor or has a vertex whose removal results in a planar graph. Therefore, if Jørgensen's conjecture holds, then Hadwiger's conjecture for $k = 5$

easily reduces to the Four Color Theorem. In 2017, Kawarabayashi, Norine, Thomas, and Wollan [35] showed that Jørgensen's conjecture holds for sufficiently large graphs.

In their work [17], Robertson, Seymour, and Thomas proved that Jørgensen's conjecture holds for each 6-connected graph in which some edge is contained in four triangles. (However, they were not able to resolve the Jørgensen conjecture. Instead, they explored different structures of a minimum counterexample to the Hadwiger conjecture.) It is natural and useful to extend this result to graphs in which some edge is contained in three triangles: Given a 6-connected graph G and triangles $a_ib_1b_2a_i$ for $i = 0, 1, 2$ in G, can we prove that G contains K_6 minor or has a vertex whose removal results in a planar graph?

A first step is to prove that 6-connected graphs are *two-three linked*: If G is a 6-connected graph and a_0, a_1, a_2, b_1, b_2 are distinct vertices of G, then G contains disjoint connected subgraphs G_1, G_2 such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$ and $\{b_1, b_2\} \subseteq V(G_2)$. In fact, Robertson and Seymour asked for a characterization of two-three linked graphs. We believe that we have such a characterization which is quite complicated (even to state) and its proof is long.

For convenience, we use $(G, a_0, a_1, a_2, b_1, b_2)$ to denote a graph G and distinct vertices a_0, a_1, a_2, b_1, b_2 of G, and call it a *rooted graph*. A *cluster* in a graph G is a set $\mathcal{X}$ of disjoint subsets of $V(G)$ such that each member of $\mathcal{X}$ induces a connected subgraph of G. We say that a rooted graph $(G, a_0, a_1, a_2, b_1, b_2)$ is *feasible* if there exists a cluster $\{X_1, X_2\}$ in G such that $\{a_0, a_1, a_2\} \subseteq X_1$ and $\{b_1, b_2\} \subseteq X_2$. We can now state our result as follows.

Theorem 1.0.1 *Let $(G, a_0, a_1, a_2, b_1, b_2)$ be a rooted graph, and assume $G + b_1b_2 + \{a_ib_j : i = 0, 1, 2$ and $j = 1, 2\}$ is 6-connected. Then $(G, a_0, a_1, a_2, b_1, b_2)$ is feasible.*

We may view the problem of characterizing feasible rooted graphs as a generalization of the following problem of characterizing 2-linked graphs: Given a graph G and four distinct vertices a_1, a_2, b_1, b_2 of G, when does G contain disjoint paths from a_1, a_2 to b_1, b_2, respectively? Several characterizations of 2-linked graphs are given in [36, 37, 38, 39] and have been used extensively in the literature for proving important structural results on graphs (e.g., in the graph minors project of Robertson and Seymour).

On a high level of the proof, we will always assume that $\gamma := (G, a_0, a_1, a_2, b_1, b_2)$ is a given rooted infeasible graph such that $b_1b_2 \notin E(G)$, $a_ib_j \notin E(G)$ for $i = 0, 1, 2$ and $j = 1, 2$, and $G^* := G + b_1b_2 + \{a_ib_j : i = 0, 1, 2 \text{ and } j = 1, 2\}$ is 6-connected.

Then in Chapter 2, we show that G has a *frame* A, B with respect to a_i for some $i \in \{0, 1, 2\}$ in $(G, a_0, a_1, a_2, b_1, b_2)$, where A, B are disjoint paths in $G - a_i$ from a_{i-1}, b_1 to a_{i+1}, b_2, respectively (with $a_{-1} = a_2, a_3 = a_0$). We say that a B-*bridge* of G is a subgraph of G induced by all edges in a component of $G - V(B)$ and all edges from that component to B. Given a frame A, B w.r.t. a_i for some $i \in \{0, 1, 2\}$, we can prove that the B-bridge of G containing a_i has a disk representation with B, a_i occurring on the boundary of the disk. Moreover, we define a *doublecross* in frame A, B, and prove that A, B has no doublecross.

These properties make the structure of G much simpler and clearer, but it is still not enough. Hence, in Chapter 3, we need to produce *good frames* and *ideal frames* A, B w.r.t. a_i for some $i \in \{0, 1, 2\}$ in G (with desired nice properties, such as the B-bridge of G containing a_i is maximal). We also divide the $(A \cup B)$-bridges of G between A and B into *slim connectors* and *fat connectors*. Then our proof is split into two cases: when there does not exist any fat connector in any ideal frame A, B, which is solved in Chapter 6, and when there exists at least one fat connector in some ideal frame A, B, which is solved in Chapter 4 and 5.

For the case without any fat connector, $G - V(A)$ has a disk representation with B and a_0 on the boundary of the disk, and any A-B path in G is induced by a single edge. So the structure of G is quite simple in some sense. For the second case, the structure is more complicated, where an A-B path in G is not just a single edge, and different A-B paths may intersect with each other. However, in both cases, we will try to find a configuration with special properties, which may help us force a small cut in G or show that $(G, a_0, a_1, a_2, b_1, b_2)$ is feasible.

Finally, we end this chapter with some notation and terminology. Let G_1, G_2 be two graphs. We use $G_1 \cup G_2$ (respectively, $G_1 \cap G_2$) to denote the graph with vertex set $V(G_1) \cup V(G_2)$ (respectively, $V(G_1) \cap V(G_2)$) and edge set $E(G_1) \cup E(G_2)$ (respectively, $E(G_1) \cap E(G_2)$). Let G be a graph, a *separation* in G is a pair (G_1, G_2) of edge-disjoint subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$. And $|V(G_1) \cap V(G_2)|$ is the *order* of the separation (G_1, G_2).

Let P be a path, and let $u, v \in V(P)$. Then $P[u, v) := P[u, v] - v$, $P(u, v] := P[u, v] - u$, and $P(u, v) := P[u, v] - \{u, v\}$. Let B be a subgraph of a graph G. Then a B-*bridge* of G is a subgraph of G induced by all edges in a component of $G - V(B)$ and all edges from that component to B.

CHAPTER 2

FRAMES

In this chapter, we state some known results and prove some lemmas that we will use. In particular, we show that an infeasible rooted graph must contain a ”frame” which consists of two disjoint paths.

A result we use often is Seymour's characterization of 2-linked graphs [37]. To state this result we introduce several concepts. A *disk representation* of a graph G is a drawing of G in a disk in which no two edges cross. A 3-*planar graph* $(G, \mathcal{A})$ consists of a graph G and a set $\mathcal{A} = \{A_1, ..., A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \varnothing$) such that

(i) for $i \neq j$, $N(A_i) \cap A_j = \varnothing$,

(ii) for $1 \leq i \leq k$, $|N(A_i)| \leq 3$, and

(iii) if $p(G, \mathcal{A})$ denotes the graph obtained from G by (for each i) deleting A_i and adding edges joining every pair of distinct vertices in $N(A_i)$, then $p(G, \mathcal{A})$ can be drawn in the plane with no edge crossings.

If, in addition, $b_0, b_1, ..., b_n$ are vertices in G such that $b_i \notin A$ for $0 \leq i \leq n$ and $A \in \mathcal{A}$, $p(G, \mathcal{A})$ can be drawn in a closed disk with no edge crossings, and $b_0, b_1, ..., b_n$ occur on the boundary of the disk in this cyclic order, then we say that $(G, \mathcal{A}, b_0, b_1, ..., b_n)$ is 3 -planar. If there is no need to specify $\mathcal{A}$, we may simply say that $(G, b_0, b_1, ..., b_n)$ is 3-planar. If $\mathcal{A} = \emptyset$, we say that $(G, b_0, b_1, ..., b_n)$ is planar. Moreover, we say that a face of (the disk representation of) G is *finite*, if the face is inside the disk.

Lemma 2.0.1 (Seymour, 1980) *Let G be a graph with distinct vertices x_1, x_2, x_3, x_4. Then either (G, x_1, x_2, x_3, x_4) is 3-planar, or G has a cluster $\{X_1, X_2\}$ such that $\{x_1, x_3\} \subseteq X_1$*

and $\{x_2, x_4\} \subseteq X_2$.

We say a sequence $(\alpha_1, \cdots, \alpha_n)$ is larger than $(\beta_1, \cdots, \beta_m)$ with respect to the lexicographic ordering if either

(i) $m < n$ and $\alpha_i = \beta_i$ for $i = 1, \cdots, m$, or

(ii) there exists j with $1 \leq j \leq min(m, n)$ so that $\alpha_j > \beta_j$ and $\alpha_i = \beta_i$ for $i = 1, \cdots, j-1$.

We will also use the following lemma to modify a certain path.

Lemma 2.0.2 *Let G be a connected graph and P be a path in G between vertices u_1 and u_2 of G, and let C denote a component of $G - V(P)$. Then one of the following holds:*

- *G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 2$, $V(P \cup C) \subseteq V(G_1)$, and $|V(G_2 - G_1)| \geq 1$, or*
- *G has an induced path Q from u_1 to u_2 such that $G - V(Q)$ is connected with $C \subseteq (G - V(Q))$.*

Proof. We choose a path Q in G from u_1 to u_2 and label the components of $G - Q$ as $C_1, \ldots, C_n$ such that $C \subseteq C_1$ and $|V(C_2)| \geq \cdots \geq |V(C_n)|$, and, subject to this, $s(Q) := (|V(C_1)|, |V(C_2)|, \cdots, |V(C_n)|)$ is maximum under the lexicographical ordering. Note that Q is well defined because of P.

Then Q is an induced path in G. For, otherwise, let Q' be the induced path in $G[Q]$ from u_1 to u_2 then $s(Q') > s(Q)$, a contradiction. If $n = 1$ then the assertion of the lemma holds. So assume $n \geq 2$.

Let $l_n, r_n \in N(C_n) \cap V(Q)$ such that $Q[l_n, r_n]$ is maximal. We may assume there exists C_j with $j < n$ such that $N(C_j) \cap P(l_n, r_n) \neq \emptyset$; otherwise, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{l_n, r_n\}$, $V(P \cup C) \subseteq V(G_1)$, and $V(C_n) \subseteq V(G_2)$, a contradiction.

Now let Q' be an induced path between u_1 and u_2 in $G[Q \cup C_n]$ such that $Q' \cap Q(l_n, r_n) = \emptyset$. Clearly, $s(Q') > s(Q)$ under the lexicographical ordering, a contradiction. □

In the remainder of this chapter, we will always assume that $\gamma := (G, a_0, a_1, a_2, b_1, b_2)$ is a given rooted graph such that $b_1b_2 \notin E(G)$, $a_ib_j \notin E(G)$ for $i = 0, 1, 2$ and $j = 1, 2$, and $G^* := G + b_1b_2 + \{a_ib_j : i = 0, 1, 2 \text{ and } j = 1, 2\}$ is 6-connected. When we write a_{i+j}, we understand that the subscript $i + j$ is taken modulo 3. In the next two lemmas, we show that G does not certain separations.

Lemma 2.0.3 *G has no separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$, $|V(G_2 - G_1)| \geq 2$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, and $(G_2, c_1, c_2, c_3, c_4, c_5, c_6)$ is planar.*

Proof. For, otherwise, let $G_2' := G_2 + \{c_1c_2, c_2c_3, c_3c_4, c_4c_5, c_5c_6, c_6c_1, c_1c_3, c_3c_5, c_5c_1\}$, which is planar as $(G_2, c_1, c_2, c_3, c_4, c_5, c_6)$ is planar.

Since G^* is 6-connected, G_2 has at least one edge from each c_i to $V(G_2 - G_1)$ and, hence, the number of edges in G_2 with at least one end in $V(G_2 - G_1)$ is at least $(6|V(G_2 - G_1)| + 6)/2 = 3|V(G_2 - G_1)| + 3 = 3|V(G_2)| - 15$. Thus, G_2' has at least $3|V(G_2)| - 15 + 9 = 3|V(G_2)| - 6$ edges.

Thus, G_2' is a planar graph with exactly $3|V(G_2')| - 6$ edges and each c_i has a unique neighbor in $G_2 - G_1$. Note that G_2' must be a planar triangulation. Therefore, the neighbors of $c_1, \cdots, c_6$ in $G_2 - G_1$ are the same. Hence, since G^* is 6-connected, $|V(G_2 - G_1)| = 1$, a contradiction. □

Lemma 2.0.4 *G has no separation (G_1, G_2) such that $|V(G_1 \cap G_2)| = 4$ and for some permutation π of $\{0, 1, 2\}$, $a_{\pi(0)}, a_{\pi(1)}, b_j \in V(G_2 - G_1)$, $|V(G_2 - G_1)| \geq 4$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, and $(G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, V(G_1 \cap G_2))$ is planar.*

Proof. Suppose to the contrary that such a separation (G_1, G_2) exists in G and let $V(G_1 \cap G_2) = \{c_1, c_2, c_4\}$ such that $(G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_4, c_3, c_2, c_1))$ is planar. Let $X := V(G_2 - G_1) - \{a_{\pi(0)}, a_{\pi(1)}, b_j\}$. Since G^* is 6-connected, we see that G_2 has at least two edges from b_j to X and at least three edges from $a_{\pi(i)}$ to X for $i \in [2]$.

Further, for any $i \in [4]$, c_i has a neighbor in X. For, otherwise, suppose, for some $i \in [4]$, c_i has no neighbor in X. Then by applying Lemma 2.0.3 to the separation $(G[V(G_1) \cup \{c_i\}], G_2 - c_i)$ in G, we see that $|X| = 1$. It then follows from planarity that b_j has at most one neighbor in X, a contradiction.

Hence, the number of edges in G_2 with at least one end in X is at least $(6|X|+1+1+1+1+3+3+2)/2 = 3|X|+6$. So $G_2' := G_2+\{c_1c_2, c_2c_3, c_3c_4, c_4a_{\pi(1)}, a_{\pi(1)}b_j, b_ja_{\pi(0)}, a_{\pi(0)}c_1, c_2a_{\pi(0)}, c_2b_j, c_2c_4, c_4b_j\}$ has edges at least $3|X| + 6 + 11 = 3(|X| + 7) - 4$. On the other hand, since G_2' is planar (as $(G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_4, c_3, c_2, c_1)$ is planar), G_2' has at most $3(|X| + 7) - 6$ edges, a contradiction. □

For $i \in \{0, 1, 2\}$, an a_i-*frame* in γ consists of disjoint paths A, B in $G - a_i$ from a_{i-1}, b_1 to a_{i+1}, b_2, respectively, such that A is induced in G, $G - V(A)$ is connected, and the B-bridge of G containing a_i does not contain A. The next lemma says that if γ is infeasible then it has a frame.

Lemma 2.0.5 *If γ is infeasible then there exists $i \in \{0, 1, 2\}$ such that γ has an a_i-frame.*

Proof. Since G^* is 6-connected, $G - \{a_0, a_1, a_2\}$ contains an induced path P from b_1 to b_2 such that $G - \{a_0, a_1, a_2\} - V(P) \neq \emptyset$. Now, by Lemma 2.0.2, $G - \{a_0, a_1, a_2\}$ has an induced path Q from b_1 to b_2 such that $C := G - \{a_0, a_1, a_2\} - V(Q)$ is connected and $C \neq \emptyset$.

Note that there exists a permutation i, j, k of $\{0, 1, 2\}$ such that $N_G(a_j) \cap V(C) \neq \emptyset$ and $N_G(a_k) \cap V(C) \neq \emptyset$, or $N_G(a_j) \cap V(C) = \emptyset$ and $N_G(a_k) \cap V(C) = \emptyset$. In the former case, $G - a_i$ contains disjoint paths from b_1, a_j to b_2, a_k, respectively. In the latter case, $N_G(a_j) \cap V(Q(b_1, b_2)) \neq \emptyset$ and $N_G(a_k) \cap V(Q(b_1, b_2)) \neq \emptyset$; so we have a path in $G[Q(b_1, b_2) + \{a_j, a_k\}]$ from a_j to a_k and a path from b_1 to b_2 in $G - \{a_0, a_1, a_2\} - V(Q(b_1, b_2))$.

Hence, there exists $i \in \{0, 1, 2\}$ such that $G - a_i$ has disjoint paths A^* and B from a_{i-1}, b_1 to a_{i+1}, b_2, respectively. Since γ is infeasible, a_i and A^* are contained in different components of $G - B$. Hence, a_i and B are contained in a component of $G - V(A^*)$. So by Lemma 2.0.2, G has an induced path A between a_{i-1} and a_{i+1} such that $G - V(A)$ is connected and $V(B) \cup \{a_i\} \subseteq V(G - A)$. Since γ is infeasible, the B-bridge of G containing a_i does not contain A. Hence, A, B is an a_i-frame in γ. □

In the next two lemmas, we derive useful information about frames in γ.

Lemma 2.0.6 *Suppose γ is infeasible and A, B is an a_i-frame in γ. Let $A_i(B)$ denote the B-bridge of G containing a_i, and let $V(A_i(B) \cap B) = \{d_1, \cdots, d_t\}$ such that $b_1, d_1, \cdots, d_t, b_2$ occur on B in this order. Then $(A_i(B) \cup B, a_i, b_1, d_1, \cdots, d_t, b_2)$ is planar.*

Proof. Let $G' = G/A$, and let a' denote the vertex representing the contraction of A. Since γ is infeasible, G' has no disjoint paths from a', b_1 to a_0, b_2, respectively. So by Lemma 2.0.1, there exists a set $\mathcal{S}$ of pairwise disjoint subsets of $V(G')$, such that $(G', \mathcal{S}, a', b_1, a_i, b_2)$ is 3-planar.

Note that for any $S \in \mathcal{S}$, $a' \in N_{G'}(S)$. For, otherwise, $N_G(S)$ is a cut in G^* separating S from $\{a_0, a_1, a_2, b_1, b_2\}$. But this contradicts the assumption that G^* is 6-connected.

Thus, for any $S \in \mathcal{S}$, we have $|N_{G'}(S) \cap V(B)| \leq 2$. Hence, $S \cap A_i(B) = \emptyset$. For otherwise, since $a' \in N_{G'}(S)$, there exists $u \in V(A_i(B) \cap B)$, such that $u \in S$. But then $G - V(A)$ contains three independent paths from u to b_1, b_2, a_i, respectively, a contradiction to the existence of cut $N_{G'}(S)$. Therefore, $A_i(B) \subseteq G' - \cup_{S \in \mathcal{S}} S$, and $G' - \cup_{S \in \mathcal{S}} S$ has a disk representation with b_1, b_2, a_i on the boundary of the disk. Thus, $A_i(B) \cup B$ inherits a disk representation with b_1, b_2, a_i occurring on the boundary of the disk. Since $A_i(B) \cup B - B$ has only one component, $(A_i(B) \cup B, a_i, b_1, d_1, \cdots, d_t, b_2)$ is planar. □

Suppose A, B is an a_i-frame in γ. Let $A_i(B)$ denote the B-bridge of G containing a_i. By a *doublecross* in A, B we mean a pair of disjoint connected subgraphs A', B' (in this order) of $G - (V(A_i(B)) \setminus V(B))$ for which there exist $a_1', a_2' \in V(A)$ and $b_1', b_2' \in V(B)$, such that $V(A')$ includes a_1', a_2' and at least one vertex of $B(b_1', b_2')$ and is otherwise disjoint from $A \cup B[b_1, b_1'] \cup B[b_2', b_2]$, and $V(B')$ includes b_1', b_2' and at least one vertex of $A(a_1', a_2')$

and is otherwise disjoint from $B \cup A[a_1, a'_1] \cup A[a'_2, a_2]$. The vertices a'_1, a'_2, b'_2, b'_1 (in this order) are called the *terminals* of the doublecross.

Lemma 2.0.7 *If γ is infeasible then there is no double cross in any frame in γ.*

Proof. Without loss of generality, assume A, B is an a_0-frame in γ. Suppose A', B' is a double cross in A, B with terminals a'_1, a'_2, b'_2, b'_1. Let $H = A(a'_1, a'_2) \cup B(b'_1, b'_2) \cup (A' - \{a'_1, a'_2\}) \cup (B' - \{b'_1, b'_2\})$. Consider the graph G' obtained from G by contracting H to a single vertex h.

Since G^* is 6-connected, then, combined with the existence of four disjoint paths $A[a_1, a'_1], A[a'_2, a_2], B[b_1, b'_1], B[b'_2, b_2]$ and Menger's theorem, G' contains five vertex disjoint paths between $\{a'_1, a'_2, b'_1, b'_2, h\}$ and $\{a_0, a_1, a_2, b_1, b_2\}$. So G contains five disjoint paths P_i, $i = 1, \ldots, 5$, (also internally disjoint from H) joining a'_1, a'_2, b'_1, b'_2 and H to $\{a_0, a_1, a_2, b_1, b_2\}$. Without loss of generality, assume that $a_1 \in V(P_1)$, $a_2 \in V(P_2)$, $b_1 \in V(P_3)$, $b_2 \in V(P_4)$, and $a_0 \in V(P_5)$.

Let $S_1 = (V(P_1 \cup P_2 \cup P_5)) \cap (\{a'_1, a'_2, b'_1, b'_2\} \cup V(H))$, and $S_2 = (V(P_3 \cup P_4)) \cap (\{a'_1, a'_2, b'_1, b'_2\} \cup V(H))$. Using the properties of a double cross, we can show that H contains a cluster $\{H_1, H_2\}$ such that $S_i \subseteq V(H_i)$, $i = 1, 2$. Let $X_1 := H_1 \cup V(P_1 \cup P_2 \cup P_5)$ and $X_2 := V(P_3 \cup P_4) \cup H_2$. Then $\{X_1, X_2\}$ is a cluster in G, a contradiction. □

In the next two lemmas, we consider the intersection of special cuts in a planar graph, which may force another cut or interesting structures of the graph.

Lemma 2.0.8 *Let γ be infeasible with an a_0-frame A, B, and let G_0 be obtained from G^* by deleting the component of $G^* - B$ containing A. Suppose (G_0, a_0, b_1, B, b_2) is planar, and G_0 has 3-cuts $\{a'_0, b'_1, b'_2\}$ and $\{a''_0, b''_1, b''_2\}$ separating $\{a_0, b_1, b_2\}$ from $B[b'_1, b'_2]$ and $B[b''_1, b''_2]$, respectively, such that $b_1, b''_1, b'_1, b''_2, b'_2, b_2$ occur on B in order, $b'_1 \neq b''_2$, and G_0 contains a path from $B(b'_1, b''_2)$ to a_0, internally disjoint from B. Then one of the following holds:*

(i) $\{b''_1, b'_2\}$ is contained in a 3-cut of G_0 separating $\{a_0, b_1, b_2\}$ from $B[b''_1, b'_2]$.

(ii) $\{b_1'', b_2'\} = \{b_1, b_2\}$*, and* $a_0' = a_0'' = a_0$.

(iii) $\{a_0'', b_1'', b_2'\} = \{a_0, b_1, b_2\}$*,* b_2'' *is a cut vertex of* G_0 *separating* b_2 *from* $\{a_0, b_1\}$*, and* a_0', a_0'', b_2', b_2'' *are incident with a common finite face of* G_0.

(iv) $\{a_0', b_1'', b_2'\} = \{a_0, b_1, b_2\}$*,* b_1' *is a cut vertex of* G_0 *separating* b_1 *from* $\{a_0, b_2\}$*, and* a_0', a_0'', b_1', b_1'' *are incident with a common finite face of* G_0.

Proof. We may assume $a_0' \neq a_0''$. For, otherwsie, since (G_0, a_0, b_1, B, b_2) is planar, either $\{a_0', b_1'', b_2'\}$ is a 3-cut in G_0 separating $\{a_0, b_1, b_2\}$ from $B[b_1'', b_2']$ and (i) holds, or $\{a_0', b_1'', b_2'\} = \{a_0, b_1, b_2\}$ and (ii) holds.

For $i \in [2]$, let F_i' be a finite face of G_0 incident with both b_i' and a_0' and let F_i'' be a finite face of G_0 incident with both b_i'' and a_0''. Since $a_0' \neq a_0''$, $b_1, b_1'', b_1', b_2'', b_2'$ occur on B in order, and G_0 contains a path from $B(b_1'', b_2')$ to a_0 and internally disjoint from B, we have $F_i' = F_i''$ for some $i \in [2]$.

By symmetry, we may assume $F_1' = F_1''$. Then a_0', a_0'', b_1', b_1'' are incident with a common finite face of G_0. Thus, either $\{a_0', b_1'', b_2'\}$ is a 3-cut of G_0 separating $\{a_0, b_1, b_2\}$ from $B[b_1'', b_2']$, or $\{a_0', b_1'', b_2'\} = \{a_0, b_1, b_2\}$ and b_1' is a cut vertex of G_0 separating b_1 from $\{a_0, b_2\}$. So (i) or (iv) holds, a contradiction. □

Lemma 2.0.9 *Let* γ *be infeasible and* A, B *be an* a_0*-frame in* γ*, and let* G_0 *be obtained from* G^* *by deleting the component of* $G^* - B$ *containing* A*. Suppose* (G_0, a_0, b_1, B, b_2) *is planar, and* G_0 *has four distinct vertices* b_1'', b_1', b_2'', b_2' *with* $b_1, b_1'', b_1', b_2'', b_2', b_2$ *on* B *in order, and* b_1'', b_2'' *are incident with a common finite face of* G_0.

(i) If $\{b_1', b_2'\}$ *is a 2-cut in* G_0 *separating* $B[b_1', b_2']$ *from* $\{a_0, b_1, b_2\}$*, then* b_1'', b_1', b_2'', b_2' *are incident with a common finite face of* G_0*, and* $\{b_1'', b_2'\}$ *is a 2-cut in* G_0 *separating* $B[b_1'', b_2']$ *from* $\{a_0, b_1, b_2\}$.

(ii) If there exists a vertex a_0' *in* G_0*, such that* $\{a_0', b_1', b_2'\}$ *is a 3-cut in* G_0 *separating* $B[b_1', b_2']$ *from* $\{a_0, b_1, b_2\}$*, then one of the following occurs:*

(a) a'_0, b''_1, b'_1, b''_2 *are incident with a common finite face of* G_0*, and* $\{a'_0, b''_1, b'_2\}$ *is a 3-cut in* G_0 *separating* $B[b''_1, b'_2]$ *from* $\{a_0, b_1, b_2\}$ *or* $\{a'_0, b''_1, b'_2\} = \{a_0, b_1, b_2\}$*;*

(b) a'_0, b''_1, b''_2, b'_2 *are incident with a common finite face of* G_0*, and* $\{b''_1, b'_2\}$ *is a 2-cut in* G_0 *separating* $B[b''_1, b'_2]$ *from* $\{a_0, b_1, b_2\}$*.*

Proof. Let F'' be a finite face of G_0 incident with b''_1, b''_2. To prove (i), we let F' be a finite face of G_0 incident with b'_1, b'_2. Since $b_1, b''_1, b'_1, b''_2, b'_2, b_2$ occur on B in order, $F' = F''$, and so (i) holds.

Next, we prove (ii). For each $i \in [2]$, we let F'_i be a finite face of G_0 incident with both b'_i and a'_0. Since $b_1, b''_1, b'_1, b''_2, b'_2, b_2$ occur on B in order, then $F'_1 = F''$ or $F'_2 = F''$. Now, if $F'_1 = F''$, then (a) of (ii) holds; if $F'_2 = F''$, then (b) of (ii) holds. □

CHAPTER 3

GOOD FRAMES AND IDEAL FRAMES

In this chapter, we show that if γ is infeasible then γ has a special frame. For an a_i-frame A, B in γ, we fix the following notation:

- $\alpha(A, B) = |\{b_i : N(b_i) \cap V(A_i(B) - a_i - B) \neq \emptyset\}|$, and
- $c(A, B) = |\{v \in V(A_i(B) \cap B) - \{b_1, b_2\} : \{v, a_i\}$ separates b_1 from b_2 in $A_i(B) \cup B\}|$.

We say that an a_i-frame A, B in γ is *good*, if among all the frames in γ,

(i) $\alpha(A, B)$ is maximum,

(ii) subject to (i), $c(A, B)$ is minimum,

(iii) subject to (ii), $A_i(B)$ is maximal.

Lemma 3.0.1 *Suppose γ is infeasible and A, B is a good frame in γ. Let $i \in \{0, 1, 2\}$ and A', B' be disjoint paths in $G - a_i$ from a_{i-1}, b_1 to a_{i+1}, b_2, respectively.*

(i) If, for some $j \in [2]$, G has a path B_0 from a_i to b_j and independent from A', B', then $\alpha(A, B) \geq 1$.

(ii) If $\{a_i, b_1, b_2\}$ is contained in a component of $G - (A' \cup (B' - \{b_1, b_2\}))$, then $\alpha(A, B) = 2$.

(iii) If G has a path B'' from b_1 to b_2 and independent from A', B', then $\alpha(A, B) = 2$ and $c(A, B) = 0$.

Proof. We first prove (i). We see that B', B_0 are contained in a common component of $G - V(A')$. By Lemma 2.0.2 and the existence of A', there exists an induced path A^* from

a_{i-1} to a_{i+1}, such that $G - V(A^*)$ is connected, and $B', B_0 \subseteq G - V(A^*)$. Since γ is infeasible, A^* and a_i are in different components of $G - B'$. So A^*, B' is a frame. By the existence of B_0, $\alpha(A^*, B') \geq 1$, and so $\alpha(A, B) \geq 1$.

Similarly, for (ii), let C be the component of $G - (A' \cup (B' - \{b_1, b_2\}))$ containing b_1, b_2, a_i, we may assume there exists an induced path A^* from a_{i-1} to a_{i+1}, such that $G - V(A^*)$ is connected, and $B', C \subseteq G - V(A^*)$. So A^*, B' is a frame. By the existence of C, $\alpha(A^*, B') = 2$, and so $\alpha(A, B) = 2$.

For (iii), since γ is infeasible, $B' \cup B'' + a_i$ must be contained in a component of $G - V(A')$. Hence, we may assume that $B'' + a_i$ is contained in a component of $G - (A' \cup (B' - \{b_1, b_2\}))$. So by (ii), $\alpha(A, B) = 2$. Now by Lemma 2.0.2 and the existence of A', there exists an induced path A^* from a_{i-1} to a_{i+1}, such that $G - V(A^*)$ is connected, and $B' \cup B'' + a_i \subseteq (G - V(A^*))$. So A^*, B' is a frame. Since $B'' + a_i$ is contained in a component of $G - (A' \cup (B' - \{b_1, b_2\}))$, we see that $c(A, B) = 0$. □

For a frame A, B in γ, an A-B *bridge* is an $(A \cup B)$-bridge of G with at least three vertices and intersecting both A and B. Let M be an A-B bridge, $l, r \in V(A \cap M)$, and $l', r' \in V(B \cap M)$, such that $A[l, r]$ and $B[l', r']$ are maximal. Then we say that l, r are the *extreme hands* of M, and that l', r' are the *feet* of M. We say that M *lies on* $B[b'_1, b'_2]$ for some $b'_1, b'_2 \in V(B)$, if $B[l', r'] \subseteq B[b'_1, b'_2]$. We say that M is *fat* if $|V(M \cap B)| \geq 2$ and *non-fat* if it is not fat.

Lemma 3.0.2 *Suppose γ is infeasible and A, B is a good a_0-frame in γ. Let $\{d_1, \cdots, d_t\} = V(B \cap A_0(B))$ such that $b_1, d_1, \cdots, d_t, b_2$ occur on B in order, and let $d_0 = b_1, d_{t+1} = b_2$. Then the following conclusions hold:*

(i) For any $i \in [t]$, $G - (A_0(B) - (B - d_i)))$ does not contain disjoint paths from a_1, b_1 to a_2, b_2, respectively.

(ii) For any A-B bridge M, $M \cap B \subseteq B[d_{i-1}, d_i]$ for some $i \in [t+1]$.

(iii) Let N be a B-bridge of G not containing A or a_0, then $|V(N \cap B)| \geq 4$, and $N \cap B \subseteq B[d_{i-1}, d_i]$ for some $i \in [t+1]$.

Proof. First, we note that (ii) and (iii) follow immediately from (i). So we prove (i). Suppose (i) fails, and let A^*, B' be disjoint paths in $G - (A_0(B) - (B - d_i)))$ from a_1, b_1 to a_2, b_2, respectively.

Then $A_0(B) \cup B'$ is contained in a component of $G - V(A^*)$. By Lemma 2.0.2 and the existence of A^*, there exists an induced path A' from a_1 to a_2, such that $G - V(A')$ is connected, and $A_0(B) \cup B' \subseteq (G - V(A'))$. So A', B' is a frame in γ. Now, due to the existence of d_i, the B-bridge of G containing a_0 is properly contained in the B'-bridge of G containing a_0, a contradiction. □

An a_i-frame A, B in γ is *ideal* if A, B is a good such that

(i) the union of those B-bridges of G not containing A or a_i is maximal,

(ii) subject to (i), the union of those fat A-B bridges is maximal,

(iii) subject to (ii), the number of non-fat A-B bridges is minimum.

Lemma 3.0.3 *Suppose γ is infeasible with ideal frame A, B. Then all A-B bridges are fat.*

Proof. Let M be a non-fat A-B bridge with extreme hands l, r and foot u. Then $V(M \cap A(l, r)) \neq \emptyset$, to avoid the cut $\{l, r, u\}$ in G^*. Note that $M - u - A(l, r)$ has a path from l to r. Hence, by Lemma 2.0.2, $M \cup A[l, r] - u$ contains an induced path P from l to r, such that $M \cup A[l, r] - u - V(P)$ is connected with $A(l, r) \subseteq M \cup A[l, r] - u - V(P)$. Let $A' := A[a_1, l] \cup P \cup A[r, a_2]$. We show that A', B contradicts the choice of A, B.

Clearly, A', B is a good frame, and the union of those B-bridges of G not containing A or a_0 is equal to the union of those B-bridges of G not containing A' or a_0. Moreover, $A(l, r)$ is contained in a non-fat A'-B bridge; otherwise, the union of those fat A'-B bridges properly contains the union of those fat A-B bridges, a contradiction.

Let $M_1, \cdots, M_k$ be the A-B bridges such that for each $i \in [k]$, $M_i \cap A(l,r) \neq \emptyset$, $M_i \neq M$. Then $k \neq 0$; otherwise, G has at least two disjoint edges from $A(l,r)$ to B (as G^* is 6-connected), which contradicts that $A(l,r)$ is contained in a non-fat A'-B bridge.

Since $M_i \cap A(l,r) \neq \emptyset$ for $i \in [k]$, $\bigcup_{i\in[k]} M_i$ and $A(l,r)$ are contained in a same non-fat A'-B bridge; so $M_1, \ldots, M_k$ are non-fat A-B bridges. Now, since $M \cup A[l,r] - u - V(P)$ is connected with $A(l,r) \subseteq M \cup A[l,r] - u - V(P)$, then $\bigcup_{i\in[k]} M_i$ and $M \cup A[l,r] - u - V(P)$ are contained in one single A'-B bridge. Hence, the number of non-fat A'-B bridges is strictly smaller than the number of non-fat A-B bridges, a contradiction. □

Let A, B be a good a_i-frame in γ, let $\{d_1, \cdots, d_t\} = V(B \cap A_i(B))$ with $b_1, d_1, \cdots, d_t, b_2$ on B in order, and let $d_0 = b_1$ and $d_{t+1} = b_2$. For any $i \in [t+1]$, we let J_i^* be the union of $B[d_{i-1}, d_i]$, all the edges between A and $B[d_{i-1}, d_i]$, all those A-B bridges M with $M \cap B \subseteq B[d_{i-1}, d_i]$, and all those B-bridges N of G with $(A + a_i) \cap N = \emptyset$ and $N \cap B \subseteq B[d_{i-1}, d_i]$. Let $u_1, u_2 \in V(A \cap J_i^*)$, such that a_1, u_1, u_2, a_2 occur on A in order with $A[u_1, u_2]$ maximal. Then we say $J_i = G[V(J_i^* \cup A[u_1, u_2])]$ is an A-B *connector*, and u_1, u_2 are the extreme hands of J_i. We say that d_{i-1}, d_i are the feet of J_i. Note that our definition does not require $J_i \cap J_j = \emptyset$ for $i \neq j$.

An A-B connector J (with feet v_1, v_2 and extreme hands u_1, u_2) is *slim* if $(J - A[u_1, u_2], B[v_1, v_2])$ is planar, and each edge of J with exactly one end in $A[u_1, u_2]$ has its other end in $B[v_1, v_2]$. Thus, no slim A-B connector contains an A-B bridge. If J is not a slim connector, we say that J is a *fat* A-B connector.

Lemma 3.0.4 *Let γ be infeasible with an ideal frame A, B. Let J be an A-B connector with feet v_1, v_2 and extreme hands u_1, u_2, such that $V(J) \backslash \{u_1, u_2, v_1, v_2\} \neq \emptyset$. Then*

(i) $u_1 \neq u_2$, there exists a unique $j \in [2]$ such that G has an A-B path from $B[b_j, v_j)$ to $A(u_1, u_2)$, and $(J - v_j, A[u_1, u_2], v_{3-j})$ is planar, and

(ii) if J is fat then $N_G(v_j) \cap V(J - v_j - A) \nsubseteq L_p$ for $p \in [2]$, where L_p denotes the subpath of the outer walk of $(J - v_j, A[u_1, u_2], v_{3-j})$ from u_p to v_{3-j} without going

through u_{3-p}.

Proof. Since $V(J)\backslash\{u_1, u_2, v_1, v_2\} \neq \emptyset$ and G^* is 6-connected, then $u_1 \neq u_2$ and G has an A-B path from $B - B[b_1, b_2]$ to $A(u_1, u_2)$. By Lemma 2.0.7, there exists a unique $j \in [2]$ such that G has an A-B path from $B[b_j, v_j)$ to $A(u_1, u_2)$.

To prove $(J - v_j, A[u_1, u_2], v_{3-j})$ is planar, let T be an A-B path from $t' \in B[b_j, v_j)$ to $t \in A(u_1, u_2)$. If $J - v_j$ contains disjoint paths A^*, B^* from u_1, t to u_2, v_{3-j}, respectively, then $A' := A[a_1, u_1] \cup A^* \cup A[u_2, a_2]$ and $B' := B[b_j, t'] \cup T \cup B^* \cup B[v_{3-j}, b_{3-j}]$ are disjoint paths in $G - v_j - V(A_0(B) - B)$ from a_1, b_1 to a_2, b_2, respectively; which contradicts (i) of Lemma 3.0.2. So assume that such A^*, B^* do not exist. Then by Theorem 2.0.1, there exist $m \geq 0$ and a set $\mathcal{D} = \{D_1, \cdots, D_m\}$ of pairwise disjoint nonempty subsets of $V(J - v_j) - \{u_1, u_2, t, v_{3-j}\}$ such that $(J - v_j, \mathcal{D}, u_1, t, u_2, v_{3-j})$ is 3-planar. We choose $D_1, \ldots, D_m$ such that $\bigcup_{i \in [m]} D_i$ is minimal. Then for all $p \in [m]$, $G[D_p \cup N_{J-v_j}(D_p)]$ does not have a disk representation with $N_{J-v_j}(D_p)$ occurring on the boundary of the disk (or else, D_p could be chosen to be empty). Obviously, $|D_p| \geq 2$.

Note that $J - v_j - A[u_1, u_2]$ is connected. For, otherwise, let C be a component of $J - v_j - A[u_1, u_2]$ disjoint from $B(v_j, v_{3-j}]$. Then $N_G(C) \subseteq V(A[u_1, u_2]) \cup \{v_j\}$. Since $G - A$ is connected, $v_j \in N(C)$; hence, $G[V(C) \cup N(C)] - E(A)$ is a non-fat A-B bridge, contradicting Lemma 3.0.3.

If $m = 0$ then $\mathcal{D} = \emptyset$, and $(J - v_j, u_1, t, u_2, v_{3-j})$ is planar; so $(J - v_j, A[u_1, u_2], v_{3-j})$ is planar as $J - v_j - A[u_1, u_2]$ is connected. Hence, $m \geq 1$. Since G^* is 6-connected, for all $p \in [m]$, $N_{J-v_j}(D_p) \cup \{v_j\}$ is not a cut of G separating D_p from other vertices. So $D_p \cap V(A) \neq \emptyset$. Since $D_p \cap \{u_1, u_2, t, v_{3-j}\} = \emptyset$, $|N_{J-v_j}(D_p) \cap A| \geq 2$. Moreover, since A is an induced path and $G[D_p \cup N_{J-v_j}(D_p)]$ does not have a disk representation with $N_{J-v_j}(D_p)$ occurring on the boundary of the disk, $D_p \not\subseteq V(A)$. Thus, $N_{J-v_j}(D_p) \not\subseteq V(A)$ as $J - v_j - A[u_1, u_2]$ is connected. So $|N_{J-v_j}(D_p)| = 3$ and $|N_{J-v_j}(D_p) \cap A| = 2$. Moreover, if we let $\{s_1, s_2, s\} = N_{J-v_j}(D_p)$ such that $s \notin V(A)$ and u_1, s_1, s_2, u_2 occur on A in order, then $J - v_j$ has a path D from s to v_{3-j} disjoint from A; or else, there exists a non-fat A-B bridge with foot v_j, or $G - V(A)$ is not connected. Moreover, since G^* is 6-connected, G has an A-B path R from $r' \in V(B - B[v_1, v_2])$ to $r \in V(A(s_1, s_2))$. By Lemma 2.0.7, $r' \in B[b_j, v_j)$.

Let $H := G[D_p \cup N_{J-v_j}(D_p)]$. If H contains disjoint paths X', R_1 from s_1, r to s_2, s, respectively, then the paths $A' := A[a_1, s_1] \cup X' \cup A[s_2, a_2]$ and $B' := B[b_j, r'] \cup R \cup R_1 \cup D \cup B[v_{3-j}, b_{3-j}]$ in $G - V(A_0(B) - B) - \{v_j\}$ from a_1, b_1 to a_2, b_2, respectively, contradict Lemma 3.0.2. So such X' and R_1 do not exist. By Lemma 2.0.1, there exist $n \geq 0$ and a set $\mathcal{V} = \{V_1, \cdots, V_n\}$ of pairwise disjoint subsets of D_p such that $(H, \mathcal{V}, s_1, r, s_2, s)$ is 3-planar. However, we see that $\{D_1, \cdots, D_m\} \backslash \{D_p\} \cup \{V_1, \cdots, V_n\}$ contradicts our choice of $\{D_1, \ldots, D_m\}$. This completes the proof of (i).

Next, we prove (ii). Since J contains disjoint paths $A[u_1, u_2]$ and $B[v_1, v_2]$, $N_G(v_j) \cap V(J - v_j - A) \neq \emptyset$. Suppose $N_G(v_j) \cap V(J - v_j - A) \subseteq L_p$ for some $p \in [2]$. Let $u \in N_G[v_j] \cap V(L_p)$, such that $u \neq u_p$, and $L_p[u_p, u]$ is minimal. Since $(J - v_j, A[u_1, u_2], v_{3-j})$ is planar, $J - v_j - A[u_1, u_2]$ is also planar. Let P' denote the subpath of the outer walk of $J - v_j - A[u_1, u_2]$ from u to v_{3-j} with $P' \subseteq L_p$. Then $N_G(v_j) \cap V(J - v_j - A) \subseteq V(P')$. Let $B' = B[b_j, v_j] \cup \{v_j u\} \cup P' \cup B[v_{3-j}, b_{3-j}]$. Then A, B' is a good frame. The union of those B-bridges of G not containing A and a_0 is contained in the union of those B'-bridges of G not containing A and a_0, which forces $B = B'$ by the choice of A, B. Moreover, by Lemma 3.0.3 and the planarity of $J - v_j$, each edge of J with exactly one end in $A[u_1, u_2]$ has its other end in $B[v_1, v_2]$; so J is a slim connector, a contradiction. □

CHAPTER 4
CORE FRAMES

In this chapter, we consider the situation when there is a fat connector for some ideal frame in γ. The first two lemmas study the inside of fat connectors, and show that each fat connector has a core in which we can find various disjoint paths.

Lemma 4.0.1 *Suppose γ is infeasible and A, B is an ideal a_0-frame in γ. Let J be a fat A-B connector with feet v_1, v_2 and extreme hands u_1, u_2, such that $(J - v_1, A[u_1, u_2], v_2)$ is planar, a_1, u_1, u_2, a_2 occur on A in order, b_1, v_1, v_2, b_2 occur on B in order, and G has an A-B path from $A(u_1, u_2)$ to $B[b_1, v_1)$. Then there exists a separation (H, L) in J of order 4 (we allow $H = J$ and L consists of u_1, u_2, v_2 and no edges), such that*

(i) *$V(H \cap L) = \{v_1, x_1, x_2, y_2\}$, u_1, x_1, x_2, u_2 occur on A in order, v_1, y_2, v_2 occur on B in order, $A[x_1, x_2] \cup B[v_1, y_2] \subseteq H$, $\{u_1, u_2, v_2\} \subseteq V(L)$;*

(ii) *$(L - A, B[y_2, v_2], v_1)$ is planar, and each edge of L with exactly one end in A has its other end in $V(B[y_2, v_2]) \cup \{v_1\}$;*

(iii) *$(H - v_1, A[x_1, x_2], y_2)$ is planar, $H - v_1 - A[x_1, x_2]$ is connected, $x_1y_2, x_2y_2 \notin E(H)$, $H - A(x_1, x_2) - \{v_1x_1, v_1x_2\}$ contains disjoint paths from v_1, y_2 to x_1, x_2, respectively, and disjoint paths from v_1, y_2 to x_2, x_1, respectively, and $V(X_1 \cap X_2) = \{y_2\}$ and $N(v_1) \cap V(H - A) \not\subseteq V(X_i)$ for $i \in [2]$, where X_i is the path from x_i to y_2 on the outer walk of $H - v_1$ without going through x_{3-i}.*

Proof. Note that by Lemma 3.0.4, if we take $H = J$ and let L consist of u_1, u_2, v_2 and no edges, then (H, L) satisfies (i) and (ii) (with $x_i = u_i$ for $i \in [2]$ and $y_2 = v_2$). Hence, we choose (H, L) satisfying (i) and (ii) and, subject to this, H is minimal. We show that (iii) holds.

Since $(J - v_1, A[u_1, u_2], v_2)$ is planar, $(H - v_1, A[x_1, x_2], y_2)$ is planar. Note that $H - v_1 - A[x_1, x_2]$ is connected; for otherwise, let C be a component of $H - v_1 - A[x_1, x_2]$ not containing y_2, which is also a component of $J - v_1 - A[u_1, u_2]$. Then either it contradicts the definition of frame that $G - V(A)$ is connected, or it contradicts Lemma 3.0.3 that all A-B bridges are fat. By the minimality of H, we see that $x_1y_2, x_2y_2 \notin E(H)$.

For $i = 1, 2$, let X_i denote the path in the outer walk of $H - v_1$ from y_2 to x_i not containing x_{3-i}. Then $V(X_1 \cap X_2) = \{y_2\}$. For, otherwise, H has a separation (H_1, H_2) such that $|V(H_1 \cap H_2)| = 1$, $y_2 \in V(H_1 - H_2)$, and $A[x_1, x_2] \subseteq H_2$. Since G^* is 6-connected, $V(H_1 - H_2) = \{y_2\}$. Let $y_2' \in V(H_1 - y_2)$. Now it is easy to check that the separation $(H - y_2, G[L + y_2'])$ contradicts the choice of (H, L) (that H is minimal).

Next we show that $N(v_1) \cap V(H - A) \not\subseteq V(X_i)$ for $i = 1, 2$. For, suppose this is false and, by symmetry, that $N(v_1) \cap V(H - A) \subseteq V(X_2)$. Let $y_2' \in N(v_1) \cap V(X_2)$ with $X_2[y_2', y_2]$ minimal. Let B' denote the path in the outer walk of $H - A$ from y_2' to y_2 not containing $X_2[y_2', y_2]$. We could choose B so that $B' \subseteq B$. However, this shows that J is not fat, a contradiction.

It remains to show that for $j \in [2]$, $H - A(x_1, x_2) - \{v_1x_1, v_1x_2\}$ contains disjoint paths from v_1, y_2 to x_{3-j}, x_j, respectively. For, otherwise, we may assume by symmetry that $H - A(x_1, x_2) - \{v_1x_1, v_1x_2\}$ does not have disjoint paths from v_1, y_2 to x_1, x_2, respectively. Hence, $H - A(x_1, x_2) - X_2 - \{v_1x_1, v_1x_2\}$ has no path from v_1 to x_1. Since $(H - v_1, A[x_1, x_2], X_2, X_1)$ is planar, there exist $x_1' \in V(A(x_1, x_2))$, $y_2' \in V(X_2)$, and a 2-separation (H_1, H_2) in $H - v_1$ such that $V(H_1 \cap H_2) = \{x_1', y_2'\}$, $x_1, y_2 \in V(H_1)$, $A[x_1', x_2] \subseteq H_2$, and $N(v_1) \cap V(H) \subseteq V(H_2 \cup A[x_1, x_2] \cup X_2)$. Then we see that the separation $(H_2, G[H_1 \cup L])$ of J contradicts the choice of (H, L). $\square$

With the notation in Lemma 4.0.1, we say that H is an A-B *core* or a *core* of the fat connector J. Moreover, we say that x_1, x_2 are the *extreme hands* of H, v_1, y_2 are the *feet* of H, and y_2 is the *main foot* of H. For convenience, we write $y_1 := v_1$. By symmetry, we may always assume that a_1, x_1, x_2, a_2 occur on A in order, and that b_1, y_1, y_2, b_2 occur on B

in order. Note that $y_1 \in V(A_0(B))$ and G has a path from a_0 to y_1 internally disjoint from B. For $i \in [2]$, let $x_i' \in V(A(x_1, x_2))$ such that x_i', x_i are incident with a common finite face of $H - y_1$, and $H - y_1$ has a path from x_i' to y_2 and internally disjoint from A. And for $i \in [2]$, let X_i' be the path from y_2 to x_i' on the outer walk of $H - \{y_1, x_i\}$ without going through x_{3-i}.

Lemma 4.0.2 *Suppose γ is infeasible, A, B is an ideal a_0-frame, and H is an A-B core with extreme hands x_1, x_2 and feet y_1, y_2, where y_2 is the main foot. Then the degree of y_2 in $H - y_1$ is at least 2 and, for $i \in [2]$, $|V(X_i(x_i, y_2))| \geq 1$ and $V(X_i \cap X_{3-i}') = \{y_2\}$. Moreover, if, for some $i \in [2]$, H does not contain disjoint paths from y_1, y_2 to x_i, x_{3-i}', respectively, and internally disjoint from A, then the following are true:*

(i) No finite face of $H - y_1$ is incident with both y_2 and a vertex of $A(x_1, x_2)$.

(ii) For any $v \in N(y_1) \cap V(H)$ with $v \notin X_{3-i}' \cup A(x_i, x_{3-i}]$, there exist $c_1 \in A(x_i, x_{3-i}')$ and $c_2 \in X_{3-i}'(x_{3-i}', y_2)$, such that $\{c_1, c_2\}$ is a cut in $H - \{y_1, x_{3-i}\}$ separating v from x_i, and there exist independent paths from v to c_1, c_2 in $H - \{y_1, x_{3-i}\}$, respectively, which are internally disjoint from $X_{3-i}' \cup A[x_i, x_{3-i}']$.

(iii) H has disjoint paths from y_1, y_2 to x_{3-i}, x_i', respectively, and internally disjoint from A.

Proof. By Lemma 4.0.1, $V(X_1 \cap X_2) = \{y_2\}$ and $x_1y_2, x_2y_2 \notin E(H)$; so the degree of y_2 in $H - y_1$ is at least 2 and $|V(X_i(x_i, y_2))| \geq 1$. Moreover, $V(X_i \cap X_{3-i}') = \{y_2\}$ for $i \in [2]$; for, suppose there exists $c \in V(X_i \cap X_{3-i}') - \{y_2\}$, then $\{c, y_1, y_2, x_{3-i}\}$ is a cut in G separating $V(X_{3-i})$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

By symmetry, we may assume that H does not contain disjoint paths from y_1, y_2 to x_1, x_2', respectively, that are internally disjoint from A.

To prove (i), suppose there exists $v_0 \in V(A(x_1, x_2))$ such that v_0, y_2 are incident with a common finite face in $H - y_1$. Since $(H - y_1, A[x_1, x_2], y_2)$ is planar, $H - y_1$ has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{y_2, v_0\}$, $X_1 \subseteq H_1$, and $X_2 \subseteq H_2$. Now, we further choose v_0 so that H_1 is minimal.

Now, we see that H_2 contains a path P_2 from y_2 to x_2' and internally disjoint from A; for otherwise, $V(H_2 \cap A) = \{x_2\}$ and, hence, $\{y_1, y_2, x_2\}$ is a cut in G separating $V(X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Now, let P_1 be the path from y_1 to x_1 in $H - V(A(x_1, x_2]) \cup \{y_2\}$ (by (iii) of Lemma 4.0.1). Since $v_0 \neq x_1$, $V(P_1 \cap H_2) = \emptyset$, and so $V(P_1 \cap P_2) = \emptyset$. However, the existence of P_1, P_2 contradicts that H does not contain disjoint paths from y_1, y_2 to x_1, x_2', respectively, and internally disjoint from A. This completes the proof of (i).

To prove (ii), let $v \in N(y_1) \cap V(H)$ such that $v \notin X_2' \cup A(x_1, x_2]$. Since $(H - \{y_1, x_2\}, A[x_1, x_2'] \cup X_2'[x_2', y_2])$ is planar and $H - y_1 - A(x_1, x_2] \cup X_2'$ does not have a path from v to x_1, there exist $c_1, c_2 \in V(A(x_1, x_2'] \cup X_2')$ such that $\{c_1, c_2\}$ is a cut in $H - \{y_1, x_2\}$ separating v from x_1. We may assume c_1, c_2 occur on $A(x_1, x_2'] \cup X_2'[x_2', y_2]$ in order.

Note that $c_1 \notin V(X_2')$, to avoid the cut $\{c_1, c_2, y_1, x_2\}$ in G^*. Moreover, $c_2 \notin A(x_2', y_2]$; or else, $H - V(A) \cup \{y_1\}$ is not connected, contradicting (iii) of Lemma 4.0.1.

We choose c_1, c_2 such that $A[c_1, x_2']$ and $X_2'[x_2', c_2]$ are minimal. Then $H - \{y_1, x_2\}$ contains independent paths from v to c_1, c_2, respectively, and internally disjoint from $A \cup X_2'$. Moreover, by (i), $c_2 \neq y_2$. This completes the proof of (ii).

To prove (iii), observe that $V(X_1' \cap X_2') = \{y_2\}$. For otherwise, let $c \in V(X_1' \cap X_2')$ with $c \neq y_2$. Since y_2 has degree at least 2 in $H - y_1$ and $x_1y_2, x_2y_2 \notin E(H)$, $\{x_1, x_2, y_1, y_2, c\}$ is a cut in G^* separating $V(X_1 \cup X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Now, let $u_2 \in V(X_2 \cap X_2')$ such that $X_2[x_2, u_2]$ is minimal. Moreover, let $v \in N(y_1) \cap V(H - A)$. If $v \in V(X_2')$ then let $P_2 = v = c_2$; and if $v \notin V(X_2')$ then by (ii), there exist $c_1 \in V(A(x_1, x_2'))$ and $c_2 \in V(X_2'(x_2', y_2))$, such that $\{c_1, c_2\}$ is a cut in $H - \{y_1, x_2\}$ separating v from x_1, and there exists a path P_2 from v to c_2 in $H - \{y_1, x_2\}$, which is internally disjoint from $X_2' \cup A[x_1, x_2']$. Since $V(X_1' \cap X_2') = \emptyset$ and $(H - y_1, A[x_1, x_2] \cup X_2)$

is planar, P_2 is disjoint from X_1'. Now, X_1' and $y_1v \cup P_2 \cup X_2'[c_2, u_2] \cup X_2[u_2, x_2]$ are disjoint paths from y_2, y_1 to x_1', x_2, respectively, in H, which are internally disjoint from A. □

The next lemma describes some interactions between cores from different connectors and finds a path B' so that A, B' is a good frame in γ which will eventually be used to form a special frame A', B' in γ.

Lemma 4.0.3 *Let γ be infeasible with an ideal a_0-frame A, B, and let H^j, $j \in [m]$, be the A-B cores in γ such that H^j has extreme hands x_1^j, x_2^j and feet y_1^j, y_2^j. Then*

(i) for any distinct $i, j \in [m]$, $A[x_1^i, x_2^i] \subseteq A[x_1^j, x_2^j]$ or $A[x_1^j, x_2^j] \subseteq A[x_1^i, x_2^i]$,

(ii) for any $j \in [m]$, $H^j - A[x_1, x_2]$ has a path P_j from y_1 to y_2 such that $|V(P_j)| \geq 3$, $H^j - P_j$ is connected, and P_j is induced in $G - y_1y_2$,

(iii) A, B' is a good a_0-frame and $A_0(B') = A_0(B)$, where B' is obtained from B by replacing $B[y_1^j, y_2^j]$ with the path P_j in (ii) for $j \in [m]$, and

(iv) with G_0' as the graph obtained from G by deleting the component of $G - B'$ containing A, $(G_0', a_0, b_1, B', b_2)$ is planar and, for any $v \in B'(y_1^j, y_2^j)$, the degree of v in G_0' is 2.

Proof. To prove (i), assume for some distinct $i, j \in [m]$ with $i \neq j$, we have $A[x_1^i, x_2^i] \not\subseteq A[x_1^j, x_2^j]$, and $A[x_1^j, x_2^j] \not\subseteq A[x_1^i, x_2^i]$. Without loss of generality, let $b_1, y_1^i, y_2^i, y_1^j, y_2^j, b_2$ occur on B in this order, and a_1, x_1^i, x_2^j, a_2 occur on A in this order with $x_2^i, x_1^j \in A(x_1^i, x_2^j)$. By Lemma 4.0.1, $H^i - A(x_1^i, x_2^i)$ has two disjoint A-B paths P_1, P_2 from y_1^i, y_2^i to x_2^i, x_1^i, respectively, and $H^j - A(x_1^j, x_2^j)$ has two disjoint A-B paths P_3, P_4 from y_1^j, y_2^j to x_2^j, x_1^j, respectively. Therefore, P_1, P_2, P_3, P_4 form a doublecross in A, B, a contradiction.

For (ii), let $j \in [m]$. Since H^j is a core, $H^j - y_1^jy_2^j - A$ has a path T_j from y_1^j to y_2^j. So by Lemma 2.0.2, $H^j - y_1^jy_2^j$ has an induced path P_j from y_1^j to y_2^j such that $H^j - y_1^jy_2^j - V(P_j)$ is connected and $A[x_1^j, x_2^j] \subseteq H^j - y_1^jy_2^j - P_j$.

To see (iii), we observe that $A_0(B')$, the B'-bridge of G containing a_0, is the same as, $A_0(B)$, the B-bridge of G containing a_0. So A, B' is also a good a_0-frame.

To prove (iv), let C denote the component of $G - B'$ containing A; so $G_0' = G - C$. By Lemma 2.0.6, $(A_0(B'), a_0, b_1, B', b_2)$ is planar. Thus, to show that $(G_0', a_0, b_1, B', b_2)$ is planar, it suffices to show that for any A-B connector J with feet v_1, v_2, $(J - C, B'[v_1, v_2])$ is planar. This is clear when J is a slim connector. So assume J is a fat connector. Then J has a separation (H, L) satisfying (i), (ii), and (iii) of Lemma 4.0.1. By (ii) of Lemma 4.0.1, $(L - A, B' \cap L)$ is planar. Since $H - B' \subseteq C$, we see that $(J - C, B'[v_1, v_2])$ is planar.

Moreover, for any $v \in B'(y_1^j, y_2^j)$, since $B'[y_1^j, y_2^j]$ is a path in the core H^j, then, combined with (ii) that P_j is induced in $G - y_1y_2$, the degree of v in G_0' is exactly 2. □

In the remainder of this chapter, suppose γ is infeasible and A, B is an ideal frame in γ. By (i) of Lemma 4.0.3, there exists an A-B core (or said an A-B' core) H with extreme hands x_1, x_2 and feet y_1, y_2 (y_2 as the main foot), such that for any core H^j with extreme hands x_1^j, x_2^j, we have $A[x_1^j, x_2^j] \subseteq A[x_1, x_2]$. We call such a core H a *main* A-B' core (or said a main A-B core). We also use B' to denote the path in (iii) of Lemma 4.0.3 and G_0' to denote the graph in (iv) of Lemma 4.0.3. By (iii) of Lemma 4.0.2, for $i \in [2]$, we let $P_{1,i}, P_{2,3-i}$ be disjoint paths in $H - A(x_1, x_2)$ from x_1, x_2 to y_i, y_{3-i}, respectively.

We consider the structure of G outside H. Let $r_1 \in V(B'[b_1, y_1))$, such that $B'[b_1, r_1)$ contains no foot of A-B cores in γ, G has no A-B' path from $A(x_1, x_2)$ to $B'[b_1, r_1)$, and subject to these conditions, $B'[b_1, r_1]$ is maximal. Then G has a path R_1 from r_1 to some $r \in V(A(x_1, x_2))$ and internally disjoint from A such that $R_1 = r_1r$ or R_1 is contained in some A-B core H' with r_1 as a foot and does not contain the other foot of H'.

For notational convenience, we let $t_1 := r_1$ and $t_2 := y_2$. We derive useful structure of G outside $A[x_1, x_2] \cup B'[t_1, t_2]$.

Lemma 4.0.4 *G has no A-B' path from $A(x_1, x_2)$ to $B' - B'[t_1, t_2]$ or from $B'(t_1, t_2)$ to $A - A[x_1, x_2]$.*

Proof. By the maximality of $B'[b_1, r_1]$, G has no A-B' path from $A(x_1, x_2)$ to $B'[b_1, t_1)$. Since no double cross exists in A, B (by Lemma 2.0.7), G has no A-B' path from $A(x_1, x_2)$ to $B'(t_2, b_2]$. Moreover, G has no A-B' path from $B'(t_1, t_2)$ to $A[a_1, x_1) \cup A(x_2, a_2]$; to avoid forming a double cross with R_1 and one of $\{P_{1,2}, P_{2,1}\}, \{P_{1,1}, P_{2,2}\}$ in A, B. $\square$

Lemma 4.0.5 *Let $e_3 = a_3b_3, e_4 = a_4b_4 \in E(G)$ with $a_3, a_4 \in V(A)$ and $b_3, b_4 \in V(B')$.*

(i) If for some $i \in [2]$, $a_3 \in V(A[a_i, x_i))$, $b_3 \in V(B'[t_2, b_2))$, $a_4 \in V(A(a_3, x_i])$, and $b_4 \in V(B'[b_1, t_1))$, then G'_0 has a 3-cut $\{a'_0, b'_1, b'_2\}$ with $b'_1 \in B'[b_1, b_4]$ and $b'_2 \in B'[t_2, b_3]$, which separates $B'[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in G'_0.

(ii) If for some $i \in [2]$, $a_3 \in V(A[a_i, x_i))$, $b_3 \in V(B'(b_1, t_1])$, $a_4 \in V(A(a_3, x_i])$, and $b_4 \in V(B'(t_2, b_2])$, then one of the following holds:

(a) G'_0 has a 3-cut $\{a'_0, b'_1, b'_2\}$ with $b'_1 \in B'[b_3, t_1]$ and $b'_2 \in B'[b_4, b_2]$, which separates $B'[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in G'_0;

(b) G'_0 has a 2-cut $\{y_1, b'_2\}$ with $b'_2 \in B'[b_4, b_2]$, which separates $B'[y_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in G'_0.

(iii) If $a_3 \in V(A[a_1, x_1])$, $a_4 \in V(A[x_2, a_2])$, and $b_3, b_4 \in V(B'(b_1, t_1))$, then G'_0 has a 3-cut $\{a'_0, b'_1, b'_2\}$ with $b'_1 \in B'[b_3, b_4]$ and $b'_2 \in B'[t_2, b_2]$, which separates $B'[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in G'_0.

(iv) If $a_3 \in V(A[a_1, x_1])$, $a_4 \in V(A[x_2, a_2])$, and $b_3, b_4 \in V(B'(t_2, b_2))$, then one of the following holds:

(a) G'_0 has a 3-cut $\{a'_0, b'_1, b'_2\}$ with $b'_1 \in B'[b_1, t_1]$ and $b'_2 \in B'[b_3, b_4]$, which separates $B'[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in G'_0;

(b) G'_0 has a 2-cut $\{y_1, b'_2\}$ with $b'_2 \in B'[b_3, b_4]$, which separates $B'[y_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in G'_0.

Proof. Suppose (i) fails. Then, since $(G_0', a_0, b_1, B', b_2)$ is planar and y_2 is the main foot of H, there exist disjoint paths B_2', A_0' in $G_0' - (B'[b_1, b_4] \cup B'[y_2, b_3])$ from b_2, a_0 to y_1, r_1, respectively. Now, $A[a_i, a_3] \cup e_3 \cup B'[y_2, b_3] \cup P_{3-i,2} \cup A(x_i, a_{3-i}] \cup R_1 \cup A_0'$ and $B'[b_1, b_4] \cup e_4 \cup A[a_4, x_i] \cup P_{i,1} \cup B_2'$ show that γ is feasible, a contradiction.

Now suppose (ii) fails. Then, since $(G_0', a_0, b_1, B', b_2)$ is planar and y_2 is the main foot of H, $G_0' - (B'[b_3, r_1] \cup B'[b_4, b_2])$ contains two disjoint paths B_1^*, A_0^* from b_1, a_0 to y_1, y_2, respectively. Now $A[a_i, a_3] \cup e_3 \cup B'[b_3, r_1] \cup R_1 \cup A(x_i, a_{3-i}] \cup P_{3-i,2} \cup A_0^*$ and $B_1^* \cup P_{i,1} \cup A[a_4, x_i] \cup e_4 \cup B'[b_4, b_2]$ show that γ is feasible, a contradiction.

If (iii) fails then, since $(G_0', a_0, b_1, B', b_2)$ is planar and y_2 is the main foot of H, $G_0' - (B'[b_3, b_4] \cup B'[t_2, b_2])$ has disjoint paths B_1^*, A_0^* from b_1, a_0 to r_1, y_1, respectively. Moreover, by Lemma 4.0.2, for some $p \in [2]$, H contains disjoint paths Y_1, Y_2 from x_p, x_{3-p}' to y_1, y_2, respectively. Thus, $A[a_1, x_1] \cup e_3 \cup B'[b_3, b_4] \cup e_4 \cup A[x_2, a_2] \cup Y_1 \cup A_0^*$ and $B_1^* \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B'[t_2, b_2]$ show that γ is feasible, a contradiction.

Finally, suppose (iv) fails. Then, since $(G_0', a_0, b_1, B', b_2)$ is planar and y_2 is the main foot of H, $G_0' - (B'[b_1, t_1] \cup B'[b_3, b_4])$ has disjoint paths B_2', A_0' from b_2, a_0 to y_2, y_1, respectively. Thus, $A[a_1, x_1] \cup e_3 \cup B'[b_3, b_4] \cup e_4 \cup A[x_2, a_2] \cup Y_1 \cup A_0'$ and $B'[b_1, r_1] \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B_2'$ show that γ is feasible, a contradiction. □

Lemma 4.0.6 *G_0' does not have 3-cuts $\{a_0', b_1', b_2\}$ and $\{a_0'', b_1, b_2''\}$ with $b_1' \in V(B'(b_1, t_1])$ and $b_2'' \in V(B'[t_2, b_2))$ such that $\{a_0', b_1', b_2\}$ separates $B'[b_1', b_2]$ from $\{a_0, b_1, b_2\}$ and $\{a_0'', b_1, b_2''\}$ separates $B'[b_1, b_2'']$ from $\{a_0, b_1, b_2\}$.*

Proof. For, suppose both 3-cuts exist. We choose $\{a_0', b_1', b_2\}$ with $B'[b_1, b_1']$ minimal, and choose $\{a_0'', b_1, b_2''\}$ with $B'[b_2'', b_2]$ minimal. Then, since G_0' has a path from a_0 to y_1 and internally disjoint from B', it follows from Lemma 2.0.8 that

(1) (ii) or (iii) or (iv) of Lemma 2.0.8 holds (and so $c(A, B') \geq 1$).

By the minimality of $B[b_1, b_1']$ and $B[b_2'', b_2]$, it follows from (1) and planarity of $(G_0', a_0, b_1, B', b_2)$ that

(2) $G_0' - B'(b_1, b_1') - B'(b_2'', b_2)$ has disjoint paths B_1^*, B_2^*, A_0^* from b_1, b_2, a_0 to b_1', b_2'', y_1, respectively, which are internally disjoint from B'.

Also by the minimality of $B[b_1, b_1']$ and $B[b_2'', b_2]$, it follows from (iii) and (iv) of Lemma 4.0.5 and Lemmas 2.0.8 and 2.0.9 that

(3) G has no edge from $B'(b_1, b_1')$ to $A[a_1, x_1]$ or no edge from $B'(b_1, b_1')$ to $A[x_2, a_2]$; and G has no edge from $B'(b_2'', b_2)$ to $A[a_1, x_1]$ or no edge from $B'(b_2'', b_2)$ to $A[x_2, a_2]$.

Next, we claim that

(4) $\alpha(A, B') \leq 1$.

For, suppose $\alpha(A, B') = 2$. Then, by (1), $a_0 = a_0' = a_0''$; so $c(A, B') \geq 2$. For convenience, let $s_1 := b_1'$ and $s_2 := b_2''$. Now, since $\alpha(A, B') = 2$, G_0' has a path A_i^* (for each $i \in [2]$) from a_0 to b_i and internally disjoint from B'. Hence, since G^* is 6-connected, $B'(b_i, s_i) \neq \emptyset$ for $i \in [2]$.

We claim that there do not exist $e = ab, e' = a'b' \in E(G)$, such that for some $i \in [2]$, $a, a' \in A(a_i, x_i)$, $b \in B'[b_1, s_1)$, and $b' \in B'(s_2, b_2]$. For, otherwise, $\alpha(A, B') = 2$ and $c(A, B') = 0$ by Lemma 3.0.1, because of the path $B'[b_1, b] \cup e \cup A[a, a'] \cup e' \cup B'[b', b_2]$ from b_1 to b_2, the path $B_1^* \cup B'[b_1', r_1] \cup R_1 \cup A[x_i, x_{3-i}) \cup P_{i,2} \cup B'[y_2, b_2''] \cup B_2^*$ from b_1 to b_2, and the path $A_0^* \cup P_{3-i,1} \cup A[x_{3-i}, a_{3-i}]$ from a_0 to a_{3-i}. This is a contradiction.

Since G^* is 6-connected, G has at least three pairwise disjoint edges from $B'(b_i, s_i)$ (for each $i \in [2]$) to $A[a_1, x_1] \cup A[x_2, a_2]$. By (3), for each $i \in [2]$, we may assume for some $j \in [2]$, G has no edge from $B'(b_i, s_i)$ to $A[a_j, x_j]$. Now, by symmetry, we assume G has no edge from $B'(b_1, s_1)$ to $A[x_2, a_2]$.

By Lemma 2.0.7, G has no cross from $A[a_1, x_1]$ to $B'(b_1, s_1)$. So, let $f_i = u_i v_i$ for $i \in [3]$ be pairwise disjoint edges of G with $u_i \in A[a_1, x_1]$ and $v_i \in B'(b_1, s_1)$, such that a_1, u_1, u_3, u_2, a_2 occur on A in order, and b_1, v_1, v_3, v_2, b_2 occur on B' in order. We choose f_1, f_2 so that $A[u_1, u_2] \cup B'[v_1, v_2]$ is maximal.

Then G has no edge from $B'(s_2, b_2)$ to $A[a_1, x_1]$. For otherwise, G has no edge from $B'(s_2, b_2)$ to $A[x_2, a_2]$ and, hence, has at least three pairwise disjoint edges from $B'(s_2, b_2)$ to $A[a_1, x_1]$. Therefore, G has an edge from $A(a_1, x_1)$ to $B'(s_2, b_2)$, which together with f_3 contradicts our claim above.

Thus, G has three pairwise disjoint edges from $B'(s_2, b_2)$ to $A[x_2, a_2]$. Since G has no cross from $A[x_2, a_2]$ to $B'(s_2, b_2)$ (by Lemma 2.0.7), we let $f_j = u_j v_j$ for $j \in \{4, 5, 6\}$ be pairwise disjoint edges of G with $u_j \in A[x_2, a_2]$ and $v_j \in B'(s_2, b_2)$, such that a_1, u_4, u_6, u_5, a_2 occur on A in order, and b_1, v_4, v_6, v_5, b_2 occur on B' in order. Choose f_4, f_5 so that $A[u_4, u_5] \cup B'[v_4, v_5]$ is maximal.

Now by the maximality of $A[u_1, u_2]$, G has an edge $f_7 = u_7 v_7$ with $u_7 \in A(u_1, u_2)$ and $v_7 \in B'[t_2, b_2]$, to avoid the cut $\{u_1, u_2, b_1, s_1, a_0\}$ in G^*. Similarly, by the maximality of $A[u_4, u_5]$, G has an edge $f_8 = u_8 v_8$ with $u_8 \in A(u_4, u_5)$ and $v_8 \in B'[b_1, t_1]$. Now, by the claim above, $v_7 \in B'[t_2, s_2]$ and $v_8 \in B'[s_1, t_1]$. Hence, f_2, f_4, f_7, f_8 form a double cross, contradicting Lemma 2.0.7. $\Box$

For $i \in [2]$, let $a'_i \in V(A[a_i, x_i])$ with $A[a_i, a'_i]$ minimal such that $a'_i = x_i$ or G has an edge from a'_i to $B'(b'_1, b_2)$. Then G has an edge $e_4 = a_4 b_4$ with $a_4 \in A(a'_1, x_1] \cup A[x_2, a'_2)$ and $b_4 \in B[b_1, b'_1)$; for, otherwise, $\{a_0, a'_1, a'_2, b'_1, b_2\}$ would be a 5-cut in G^* separating H from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. By symmetry, we may assume

(5) $a_4 \in A(a'_1, x_1]$.

Let $e_3 = a_3 b_3 \in E(G)$ with $a_3 = a'_1$ and $b_3 \in B'(b'_1, t_1] \cup B'[t_2, b_2)$. Since e_3, e_4 and the paths in H do not form a double cross (by Lemma 2.0.7), we have

(6) $b_3 \in B'[t_2, b_2)$.

Let $e = ab \in E(G)$ with $a \in A[a_1, a_3]$ and $b \in B'[b_3, b_2]$, such that $B'[b, b_2]$ is minimal, and subject to this, $A[a_1, a]$ is minimal. Further, let $e' = a'b' \in E(G)$ with $a' \in A[a_1, a_4]$ and $b' \in B'[b_1, b_4]$, such that $B'[b_1, b']$ is minimal, and subject to this, $A[a_1, a']$ is minimal.

Similarly, for each $i \in [2]$, let $a_i'' \in V(A[a_i, x_i])$ with $A[a_i, a_i'']$ minimal such that $a_i'' = x_i$ or G has an edge from a_i'' to $B'(b_1, b_2'')$. Since G^* is 6-connected, there exist $j \in [2]$ and $e_6 = a_6b_6 \in E(G)$ such that $a_6 \in A(a_j'', x_j]$ and $b_6 \in B'(b_2'', b_2]$. Since $a_j'' \neq x_j$, it follows from Lemma 2.0.7 that there exists $e_5 = a_5b_5 \in E(G)$ such that $a_5 = a_j''$ and $b_5 \in B'(b_1, t_1]$.

(7) $b \in B'(b_2'', b_2]$.

For, otherwise, $b \notin B'(b_2'', b_2]$. Then, $j = 2$ and $a_6 \in A[x_2, a_2'')$ by the choice of e. Hence, $b_5 \in B'[b_1, b_4]$ to avoid the double cross e_3, e_4, e_5, e_6. So $b_5 = b_1$ by (3), and thus $a_5 \neq a_2$. Let $e_2^* = a_2b_2^* \in E(G)$. Then $b_2^* \in B'[b_6, b_2]$ to avoid the double cross e_5, e_2^*, e_3, e_6.

Note that $a_5 \neq x_2$. Then $\alpha(A, B') = 2$ by Lemma 3.0.1 and the following paths: the path $A[a_5, a_2] \cup e_5$ from a_2 to b_1, the path $e_2^* \cup B'[b_2^*, b_2]$ from a_2 to b_2, the path $B_1^* \cup B'[b_1', r_1] \cup R_1 \cup A(x_1, x_2] \cup P_{2,2} \cup B'[y_2, b_2''] \cup B_2^*$ from b_1 to b_2, and the path $A_0^* \cup P_{1,1} \cup A[a_1, x_1]$ from a_0 to a_1. This is a contradiction to (4). □

If $a' \neq x_1$ then $\alpha(A, B') = 2$ by Lemma 3.0.1 and the following paths: the path $A[a_1, a'] \cup e' \cup B'[b_1, b']$ from a_1 to b_1, the path $A[a_1, a] \cup e \cup B'[b, b_2]$ from a_1 to b_2, the path $B_1^* \cup B'[b_1', r_1] \cup R_1 \cup A[x_1, x_2) \cup P_{1,2} \cup B'[y_2, b_2''] \cup B_2^*$ from b_1 to b_2, and the path $A_0^* \cup P_{2,1} \cup A[x_2, a_2]$ from a_0 to a_2. This contradicts (4).

So $a' = x_1$. Hence, by the choice of e' and Lemma 2.0.7, G has no edge from $A[a_1, x_1)$ to $B'[b_1, t_1]$. Thus, G has an edge from a_1 to $B'[t_2, b_2]$. So by the choice of e and by Lemma 2.0.7, $a = a_1$ and, hence, $b \neq b_2$.

We claim $a_6 \in A[x_2, a_2'')$. For, otherwise, $a_6 \in A(a_1'', x_1]$. Then $a_5 \in A[a_1, x_1)$. Now, e_5 contradicts the choice of e', or $e_5, e', P_{1,2}, P_{2,1}$ form a double cross, contradicting Lemma 2.0.7.

Thus, by (3), $b_6 = b_2$. So $b_5 \in B'[b_1, b']$ to avoid the double cross e, e', e_5, e_6.

Suppose H contains disjoint paths Y_1, Y_2 from x_1, x_2' to y_1, y_2, respectively, and internally disjoint from A. Then $\alpha(A, B') = 2$ by Lemma 3.0.1 and the following paths: the path $A[a_5, a_2] \cup e_5 \cup B'[b_1, b_5]$ from a_2 to b_1, the path $A[a_6, a_2] \cup e_6 \cup B'[b_6, b_2]$ from a_2 to b_2, the path $B_1^* \cup B'[b_1', r_1] \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B'[y_2, b_2''] \cup B_2^*$ from b_1 to b_2, and the path $A_0^* \cup Y_1 \cup A[a_1, x_1]$ from a_0 to a_1. This contradicts (4).

So by Lemma 4.0.2, H has disjoint paths Y_1, Y_2 from x_2, x_1' to y_1, y_2, respectively, and internally disjoint from A. We have a contradiction to (4) as $\alpha(A, B') = 2$ because of Lemma 3.0.1 and the following paths: the path $A[a_1, x_1] \cup e' \cup B'[b_1, b']$ from a_1 to b_1, the path $A[a_1, a] \cup e \cup B'[b, b_2]$ from a_1 to b_2, the path $B_1^* \cup B'[b_1', r_1] \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B'[y_2, b_2''] \cup B_2^*$ from b_1 to b_2, and the path $A_0^* \cup Y_1 \cup A[x_2, a_2]$ from a_0 to a_2. □

Lemma 4.0.7 *Let $\{a_0', b_1', b_2'\}$ be a cut in G_0' separating $B'[b_1', b_2']$ from $\{a_0, b_1, b_2\}$, with $b_1' \in B'[b_1, t_1]$ and $b_2' \in B[t_2, b_2]$. Then $b_1' = b_1$, $b_2' \neq b_2$, $a_0' = a_0$, y_1 is a cut vertex in G_0' separating b_2 from $\{a_0, b_1\}$, b_2 has degree 1 in G_0', and for some $p \in [2]$, G has an edge from b_2 to x_p and no edge from b_2 to $A - x_p$.*

Proof. For $i \in [2]$, let $a_i' \in V(A[a_i, x_i])$ with $A[a_i, a_i']$ minimal such that $a_i' = x_i$ or G has an edge from a_i' to $B'(b_1', b_2)$. Since G^* is 6-connected, there exist $i, j \in [2]$ such that G has an edge $e_4 = a_4 b_4$ with $a_4 \in A(a_i', x_i]$ and $b_4 \in B'[b_j, b_j')$. By symmetry, assume $i = 1$. Then $a_1' \neq x_1$ and let $e_3 = a_3 b_3 \in E(G)$ such that $a_3 = a_1'$ and $b_3 \in B'(b_1', t_1] \cup B'[t_2, b_2')$. Now $b_3 \in B'[t_{3-j}, b_{3-j}')$, to avoid the double cross formed by e_3, e_4 and two paths in H (by Lemma 2.0.7).

First, we show that

(1) $b_1' = b_1$.

For, suppose $b_1' \neq b_1$. Choose the 3-cut $\{a_0', b_1', b_2'\}$ with $b_1' \neq b_1$, such that $B[b_2', b_2]$ is minimal and, subject to this, $B[b_1, b_1']$ is minimal.

Observe that $b_4 \in B[b_1, b_1')$. For, otherwise, $b_4 \in B(b_2', b_2]$. Then $b_3 \in B(b_1', t_1]$. Now, by Lemma 2.0.9 and (ii) of Lemma 4.0.5, G_0' has a 3-cut contradicting the choice of $\{a_0', b_1', b_2'\}$.

Then $b_3 \in B'[t_2, b'_2)$. Hence, because of e_3, e_4, it follows from (i) of Lemma 4.0.5 that G'_0 has a 3-cut $\{a''_0, b''_1, b''_2\}$ with $b''_1 \in B'[b_1, b_4]$ and $b''_2 \in B'[t_2, b_3]$, separating $B'[b''_1, b''_2]$ from $\{a_0, b_1, b_2\}$. By Lemma 2.0.8 and the choice of $\{a'_0, b'_1, b'_2\}$, we have $b''_1 = b_1$.

By Lemma 4.0.6, $b'_2 \neq b_2$. Hence, by Lemma 2.0.8, there exists $a^*_0 \in V(G'_0)$, such that $\{b''_1, b'_2, a^*_0\}$ is a 3-cut in G'_0 separating $\{a_0, b_1, b_2\}$ from $B'[b''_1, b'_2]$. For $i \in [2]$, let $a''_i \in A[a_i, x_i]$ with $A[a_i, a''_i]$ minimal such that $a''_i = x_i$ or G has an edge from a''_i to $B'(b''_1, b'_2)$.

Since G^* is 6-connected, there exist $k \in [2]$ and $e_5 = a_5b_5 \in E(G)$ with $a_5 \in A(a''_k, x_k]$ and $b_5 \in B'(b'_2, b_2]$. Let $e_6 = a_6b_6 \in E(G)$ with $a_6 = a''_k$ and $b_6 \in B'(b''_1, t_1] \cup B'[t_2, b'_2)$. Then $b_6 \in B'(b''_1, t_1]$, to avoid the double cross formed by e_5, e_6 and two paths in H. Because of e_5 and e_6, it follows from (ii) of Lemma 4.0.5 and the choice of $\{a'_0, b'_1, b'_2\}$ that G'_0 has a 2-cut $\{y_1, b^*_2\}$ with $b^*_2 \in B'[b_5, b_2]$, separating $B'[y_1, b^*_2]$ from $\{a_0, b_1, b_2\}$. But then, by Lemma 2.0.9, $\{y_1, b^*_2\}$ and $\{a'_0, b'_1, b'_2\}$ force a 3-cut in G'_0, which contradicts the choice of $\{a'_0, b'_1, b'_2\}$. □

Since G^* is 6-connected, it follows from (1) that $b_2 \neq b'_2$. We choose $\{a'_0, b'_1, b'_2\}$ so that $B[b_2, b'_2]$ is minimal. Then, by (1) and (ii) of Lemma 4.0.5, G'_0 has a 2-cut $\{y_1, b''_2\}$ with $b''_2 \in B'[b_4, b_2]$, separating $B'[y_1, b''_2]$ from $\{a_0, b_1, b_2\}$.

Moreover, $b''_2 = b_2$; for, otherwise, by Lemma 2.0.9, $\{y_1, b''_2\}$ and $\{a'_0, b'_1, b'_2\}$ force a 3-cut in G'_0, which contradicts the choice of $\{a'_0, b'_1, b'_2\}$. Hence, y_1 is a cut vertex in G'_0 separating b_2 from $\{a_0, b_1\}$ and $\alpha(A, B') \leq 1$. And (for any choice of $\{a'_0, b'_1, b'_2\}$,) $a'_0 = a_0$; or else, since y_1 is a cut vertex in G'_0 separating b_2 from $\{a_0, b_1\}$, $\{b_1, a'_0, b'_2, b_2\}$ is a cut in G separating a_0 from $\{a_1, a_2\}$, a contradiction.

So by (1), $G'_0 - V(B'(b_1, t_1) \cup B'(y_1, b_2])$ has disjoint paths B^*_1, A^*_0 from b_1, a_0 to t_1, y_1, respectively, such that A^*_0 is internally disjoint from B'. By the choice of $\{a'_0, b'_1, b'_2\}$, $G'_0 - V(B'(b'_2, b_2))$ has a path B^*_2 from b_2 to b'_2.

(2) For $i \in [2]$, if G has an edge from $B'(b'_2, b_2]$ to $A[a_i, x_i)$, then G has no edge from $A[a_i, x_i)$ to $B'[b_1, t_1)$.

For, suppose for some $i \in [2]$, G has an edge e from $b \in B'(b_2', b_2]$ to $a \in A[a_i, x_i)$ and an edge e' from $a' \in A[a_i, x_i)$ to $b' \in B'[b_1, t_1)$.

Then, $\alpha(A, B') = 2$, by Lemma 3.0.1 and the following paths: $A[a_i, a'] \cup e' \cup B'[b_1, b']$ from a_i to b_1, the path $A[a_i, a] \cup e \cup B'[b, b_2]$ from a_i to b_2, the path $B_1^* \cup R_1 \cup A[x_i, x_{3-i}) \cup P_{i,2} \cup B_2^*$ from b_1 to b_2, and the path $A_0^* \cup P_{3-i,1} \cup A[x_{3-i}, a_{3-i}]$ from a_0 to a_{3-i}. This is a contradiction. $\square$

(3) $B'(b_2', b_2) = \emptyset$, and so b_2 has degree 1 in G_0'.

For, suppose $B'(b_2', b_2) \neq \emptyset$. Then, as G^* is 6-connected, G has edges from $B'(b_2', b_2)$ to $A[a_1, x_1] \cup A[x_2, a_2]$.

Indeed, G has an edge e_3 from $B'(b_2', b_2)$ to $A[a_1, x_1]$, and an edge e_4 from $B'(b_2', b_2)$ to $A[x_2, a_2]$. For otherwise, there exists $i \in [2]$, such that all edges of G from $B'(b_2', b_2)$ to A end in $A[a_i, x_i]$. Let $u_1, u_2 \in V(A[a_i, x_i])$, such that G has edges from $B'(b_2', b_2)$ to u_1, u_2, respectively, and, subject to this, $A[u_1, u_2]$ is maximal. Now, by Lemma 2.0.7, G has no edge from $A(u_1, u_2)$ to $B'[t_2, b_2')$. Moreover, by (2), G has no edge from $A(u_1, u_2)$ to $B'[b_1, t_1)$. But then, $\{t_1, u_1, u_2, b_2', b_2\}$ is a cut in G separating $V(A[u_1, u_2] \cup B'[b_2', b_2])$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Now $A[a_1, x_1] \cup e_3 \cup B'(b_2', b_2) \cup e_4 \cup A[x_2, a_2] \cup Y_1 \cup A_0^*$ and $B'[b_1, r_1] \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B'[y_2, b_2'] \cup B_2^*$ show that γ is feasible, a contradiction. $\square$

(4) G has no edge from b_2 to $A[a_1, x_1) \cup A(x_2, a_2]$.

Suppose for some $i \in [2]$, G has an edge e from b_2 to $a \in A[a_i, x_i)$. Let $e' = a_1 b' \in E(G)$ with $b' \neq t_1$. Obviously, $b' \notin B'[t_2, b_2)$; otherwise, e, e' and two disjoint paths in H force a double cross, contradicting Lemma 2.0.7.

So $b' \in B[b_1, t_1)$. Now $\alpha(A, B') = 2$ by Lemma 3.0.1 and the following paths: the path $e' \cup B'[b_1, b']$ from a_i to b_1, the path $A[a_i, a] \cup e$ from a_i to b_2, the path $B_1^* \cup R_1 \cup A[x_i, x_{3-i}) \cup P_{i,2} \cup B'[y_2, b_2]$ from b_1 to b_2, and the path $A_0^* \cup P_{3-i,1} \cup A[x_{3-i}, a_{3-i}]$ from a_0 to a_{3-i}. However, this is a contradiction. $\square$

Now, since the degree of b_2 in G is at least 2, it follows from (4) that G has an edge from b_2 to x_p for some $p \in [2]$. If G has no edge from b_2 to x_{3-p} then we are done. So assume $b_2x_1, b_2x_2 \in E(G)$. Then $a_1 \neq x_1$ and $a_2 \neq x_2$. Now, by Lemma 2.0.7, G has no edge from $\{a_1, a_2\}$ to $B'[t_2, b_2)$. Since G^* is 6-connected, G has edges e_1, e_2 from $B'(b_1, t_1)$ to a_1, a_2, respectively. But then, it follows from (iii) of Lemma 4.0.5 that G_0' contains a 3-cut, which contradicts (1). □

Lemma 4.0.8 *H is the unique main A-B' core in γ.*

Proof. Suppose for a contradiction that H'' is a main A-B' core with $H'' \neq H$, and let w_1, w_2 be the feet of H'' (with w_2 as the main foot). Then, by Lemma 2.0.7, $w_2 = r_1$ and $b_1, w_2, w_1, y_1, y_2, b_2$ occur on B' in order.

Recall that the definition of x_i', X_i' before Lemma 4.0.2. For $i \in [2]$, let $x_i'' \in V(A(x_1, x_2))$ such that x_i'', x_i are incident with a common finite face of $H'' - w_1$, and $H'' - w_1$ has a path from x_i'' to w_2 and internally disjoint from A. So for $i \in [2]$, let X_i'' be the path from w_2 to x_i'' on the outer walk of $H'' - \{w_1, x_i\}$ without going through x_{3-i}, and, moreover, let X_i^* be the path from x_i to w_2 on the outer walk of $H'' - w_1$ without going through x_{3-i}.

Suppose H contains disjoint paths from y_1, y_2 to x_2, x_1', respectively, and internally disjoint from A, as well as disjoint paths from y_1, y_2 to x_1, x_2', respectively, and internally disjoint from A. Then, by Lemma 2.0.7, for any $i \in [2]$, H'' does not contain disjoint paths from w_1, w_2 to x_i, x_{3-i}'', respectively, and internally disjoint from A. This contradicts (iii) of Lemma 4.0.2.

Hence, by symmetry, we may assume that H contains no disjoint paths from y_1, y_2 to x_1, x_2', respectively, and internally disjoint from A. Then by Lemma 4.0.2, H contains disjoint paths Y_1', Y_2' from y_1, y_2 to x_2, x_1', respectively, and internally disjoint from A.

Then by Lemma 2.0.7 and 4.0.2, we may further assume H'' contains disjoint paths Y_1'', Y_2'' from w_1, w_2 to x_2, x_1'', respectively, and internally disjoint from A, but no disjoint paths from w_1, w_2 to x_1, x_2'', respectively, and internally disjoint from A. Moreover, by

(i) of Lemma 4.0.2, $H - \{y_1, y_2\} \cup V(A(x_1, x_2))$ contains a path D' from x_1 to x_2, and $H'' - \{w_1, w_2\} \cup V(A(x_1, x_2))$ contains a path D'' from x_1 to x_2.

(1) There is no A-B' path in G from $A(x_1, x_2)$ to $B'(w_1, y_1)$.

For, suppose that P is an A-B' path from $p \in V(A(x_1, x_2))$ to $p' \in V(B'(w_1, y_1))$. Then $G'_0 - B'(w_2, w_1) - B'[y_2, b_2]$ does not contain disjoint paths B_1^*, A_0^* from b_1, a_0 to p', y_1, respectively; otherwise, $A[a_1, x_1] \cup D'' \cup A[x_2, a_2] \cup Y'_1 \cup A_0^*$ and $B_1^* \cup P \cup A(x_1, x_2) \cup Y'_2 \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction. Hence, there exists $w' \in V(B'(w_2, w_1))$, $a'_0 \in V(G'_0)$, and $b'_2 \in V(B'[y_2, b_2])$, such that $\{w', a'_0, b'_2\}$ is a 3-cut in G'_0 separating $B'[w', b'_2]$ from $\{a_0, b_1, b_2\}$.

Now $b_1 = w_2$. For, suppose not. Since w_1, w_2 are feet of H'', w_1, w_2 are incident with a common finite face of G'_0. Therefore, $\{w_2, a'_0, b'_2\}$ is a 3-cut in G'_0 separating $B'[w_2, b'_2]$ from $\{a_0, b_1, b_2\}$, a contradiction to Lemma 4.0.7. Similarly, by the symmetry between H and H'', we can also prove $b_2 = y_2$.

Now, since $b'_2 \in V(B'[y_2, b_2])$, $b'_2 = b_2$. So $a'_0 = a_0$; or else, $\{b_1, a'_0, b_2\}$ is a 3-cut in G'_0 separating a_0 from $B'(b_1, b_2)$, a contradiction. Then a_0, b_1, w', w_1 are incident with a common finite face of G'_0. Similarly, by the symmetry between H and H'', a_0, b_2, y_1 are incident with a common finite face of G'_0, which implies $\alpha(A, B') = 0$.

By Lemma 4.0.2, $V(X''_2 \cap X_1^*) - \{w_2\} = \emptyset$. Now $\alpha(A, B') \geq 1$ by Lemma 3.0.1 and the following paths: the path $A_0^* \cup Y'_1 \cup A[x_2, a_2]$ from a_0 to a_2, the path $X''_2 \cup A(x_1, x_2) \cup Y'_2$ from b_1 to b_2, and the path $A[a_1, x_1] \cup X_1^*$ from a_1 to b_1. This is a contradiction. □

(2) $a_1 = x_1$ and $a_2 = x_2$.

Recall that for $i \in [2]$, $P_{1,i}$ and $P_{2,3-i}$ are disjoint paths from x_1, x_2 to y_i, y_{3-i}, respectively, in $H - A(x_1, x_2)$. For $i \in [2]$, let $Q_{1,i}, Q_{2,3-i}$ be disjoint paths from x_1, x_2 to w_i, w_{3-i}, respectively, in $H'' - A(x_1, x_2)$.

We claim that for $i \in [2]$, G has no edge from $A[a_i, x_i)$ to $B'(b_1, w_2]$. For, suppose there exists $e' = a'b' \in E(G)$ with $a' \in A[a_i, x_i)$ and $b' \in B'(b_1, w_2]$. Then $b_1 \neq w_2$.

By Lemma 4.0.7, $G'_0 - B'[b', w_2] - B'[y_2, b_2]$ contains disjoint paths B_1^*, A_0^* from b_1, a_0 to w_1, y_1, respectively. Now $A[a_i, a'] \cup e' \cup B'[b', w_2] \cup Q_{3-i,2} \cup A[x_{3-i}, a_{3-i}] \cup P_{3-i,1} \cup A_0^*$ and $B_1^* \cup Q_{i,1} \cup P_{i,2} \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction.

Due to the symmetry between H and H'', with the same argument above, we can show that for $i \in [2]$, G has no edge from $A[a_i, x_i)$ to $B'[y_2, b_2)$. Thus, (2) follows from Lemma 4.0.4 and the assumption that G^* is 6-connected. □

(3) $H'' - V(X_1^* \cup X_2^*)$ contains a path Q'' from w_1 to $A(x_1, x_2)$; and $H - V(X_1 \cup X_2)$ contains a path Q from y_1 to $A(x_1, x_2)$.

By the symmetry between H and H'', we only prove the existence of Q''. Suppose for a contradiction that Q'' does not exist.

We see that $(N(w_1) \cap V(H'')) \subseteq V(X_2'' \cup A(x_1, x_2])$. For, otherwise, by (ii) of Lemma 4.0.2, there exists $v'' \in N(w_1) \cap V(H'')$, $c_1'' \in A(x_1, x_2'')$, and $c_2'' \in X_2''(x_2'', w_2)$, such that $v'' \notin X_2'' \cup A(x_1, x_2]$, $\{c_1'', c_2''\}$ is a cut in $H'' - \{w_1, x_2\}$ separating v'' from x_1, and there exists a path P_1'' from v'' to c_1'' in $H'' - w_1 - x_2$, which is internally disjoint from $X_2'' \cup A[x_1, x_2'']$. But then, $w_1 v'' \cup P_1''$ is a path from w_1 to $A(x_1, x_2)$ in $H'' - V(X_1^* \cup X_2^*)$, a contradiction.

Now, since Q'' does not exist, combined with $(N(w_1) \cap V(H'')) \subseteq V(X_2'' \cup A(x_1, x_2])$, we may further assume $(N(w_1) \cap V(H'')) \subseteq V(X_2^*)$, contradicting (iii) of Lemma 4.0.1. □

(4) $b_1 = w_2$ and $b_2 = y_2$.

By the symmetry between H and H'', we only show $b_1 = w_2$. Suppose for a contradiction that $b_1 \neq w_2$.

Since w_1, w_2 are incident with a common finite face of G'_0, it follows from Lemma 4.0.7 that $G'_0 - B'[w_2, w_1) - B'[y_2, b_2]$ contains disjoint paths B_1^*, A_0^* from b_1, a_0 to w_1, y_1, respectively.

Now, $A[a_1, x_1] \cup X_1^* \cup X_2^* \cup A[x_2, a_2] \cup Y_1' \cup A_0^*$ and $B_1^* \cup Q'' \cup A(x_1, x_2) \cup Y_2' \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction. □

Note that G has no A-B' path from a_1 to $B'(w_1, y_1)$, as such a path together with Y_2'', Y_1', Y_2' forms a double cross, contradicting Lemma 2.0.7. So by (1) and (4), $\{b_1, b_2, w_1, y_1, a_2\}$ is a cut in G separating a_0 from a_1, a contradiction. □

We now use A, B' to form a new frame A', B', called *core* frame.

Lemma 4.0.9 *Let M_0 denote the union of all the A-B' bridges that are disjoint from $H - A - y_1$. Then there exists an induced path $A' \subseteq (A \cup M_0) - B'$ from a_1 to a_2 in G, such that $A'[a_i, x_i] = A[a_i, x_i]$ for $i \in [2]$ and the following hold:*

(i) *A', B' is a good frame in γ.*

(ii) *Each A'-B' bridge lying on $B'[r_1, y_1]$ is contained in some A-B' bridge.*

(iii) *There exists an induced subgraph H^* in G, such that $A'[x_1, x_2] \cup H \subseteq H^*$, all A'-B' bridges not lying on $B'[r_1, y_1]$ are contained in H^*, and H^* is separated from $\{a_0, a_1, a_2, b_1, b_2\}$ by $V(A'[x_1, x_2]) \cup \{y_1, y_2\}$ in G.*

(iv) *For any $v \in (V(H^*) - V(A') \cup \{y_1\})$, $H^* - y_1$ contains a path from v to y_2 and internally disjoint from A'.*

(v) *If l, r are the extreme hands of an A'-B' bridge lying on $B'[r_1, y_1]$ then $\{l, r\} \neq \{x_1, x_2\}$, and $H^* - y_1$ does not contain a path from y_2 to $A'(l, r)$ and internally disjoint from A'.*

Proof. We choose the induced path A' so that $A' \subseteq A \cup M_0 - B'$ is from a_1 to a_2, such that $A'[a_i, x_i] = A[a_i, x_i]$ for $i \in [2]$, (i)-(iv) are satisfied, and, subject to this, H is maximal. Note that such A' exists, as A satisfies (i)-(iv).

To prove (v), let M be an A'-B' bridge M lying on $B'[r_1, y_1]$ with extreme hands l, r and feet l', r'. If $\{l, r\} = \{x_1, x_2\}$ then, since M is contained in an A-B' bridge (by (ii)), M

is contained in a main A-B' core, a contradiction to Lemma 4.0.8. Hence, $H - y_1$ contains a path Y_2 from y_2 to $y_2' \in A'(l,r)$ and internally disjoint from A'.

Let T be an induced path in $M - V(A'(l,r) \cup B'[l',r'])$ from l to r, and let $C_1, C_2, \ldots, C_n$ be the components of $M \cup A'[l,r] \cup B'[l',r'] - V(T)$ not containing $A'(l,r)$ and not containing $B'[l',r']$. We choose T, such that $|T| := (-|V(\bigcup_{i\in[n]} C_i)|, |V(C_1)|, |V(C_2)|, \ldots, |V(C_n)|)$ is maximal with respect to the lexicographical ordering.

We claim $n = 0$. For, suppose $n > 0$. Let $l_n, r_n \in N(C_n) \cap V(T)$ such that $T[l_n, r_n]$ is maximal. Since G^* is 6-connected, there exists another component C of $(M \cup A'[l,r] \cup B'[l',r']) - V(T)$, such that $N(C) \cap T(l_n, r_n) \neq \emptyset$. Now, let T' be an induced path in $G[T \cup C_n]$ between l_n and r_n, such that $T' \cap T(l_n, r_n) = \emptyset$. Clearly, $|T'| > |T|$, a contradiction.

Now, let A'' be obtained from A' by replacing $A'[l,r]$ with T. Clearly, $A''[a_i, x_i] = A[a_i, x_i]$ for $i \in [2]$. Since T is induced, A'' is induced. Moreover, since $n = 0$, then any component of $G[V(M \cup A'[l,r] \cup B'[l',r'])] - T$ contains $A'(l,r)$ or $B'[l',r']$, and so $G - V(A'')$ is connected. Hence, A'', B' is a frame. Since $A_0''(B') = A_0'(B') = A_0(B')$, we see that A'', B' is a good frame in γ.

Next, we show that G has no A'-B' path from $A'(l,r)$ to $B'[b_1, y_1)$ and disjoint from T. For otherwise, let S be an A'-B' path from $s \in A'(l,r)$ to $s' \in B'[b_1, y_1)$ and disjoint from T. Then A'' and $B'[b_1, s'] \cup S \cup A'[s, y_2'] \cup Y_2 \cup B'[y_2, b_2]$ are disjoint paths from a_1, b_1 to a_2, b_2, respectively, in $G - V(A_0(B') - B') - y_1$, a contradiction to (i) of Lemma 3.0.2.

Hence, there does not exist an A'-B' bridge N lying on $B'[r_1, y_1]$, such that $N \neq M$, $N \cap A'(l,r) \neq \emptyset$, and $N \cap B'[b_1, y_1) \neq \emptyset$. So each A''-B' bridge lying on $B'[r_1, y_1]$ must be contained in some A'-B' bridge and, hence, contained in some A-B' bridge. So A'', B' satisfies (ii).

Moreover, $V(A''[x_1, x_2]) \cup \{y_1, y_2\}$ is a cut in G separating $V(H)$ from $\{a_0, a_1, a_2, b_1, b_2\}$. Now, we let V'' be the set of vertices of $A'' \cup B'[b_1, y_1] \cup B'[y_2, b_2]$-bridge of G containing $A'(l,r)$, and let $H'' := G[V'' \cup V(A''[x_1, x_2])]$. Then clearly (iii) and (iv) holds for A'', B'. However, H'' properly contains H, a contradiction. □

CHAPTER 5

INSIDE THE MAIN A'-B' CORE

We use the notation of the previous chapter: γ is infeasible, A', B' is a core frame, and let $H' := H^* - \{x_1y_2, x_2y_2\}$ with extreme hands x_1, x_2 and feet y_1, y_2 (such that y_2 is the main foot), where B' is defined as in Lemma 4.0.3, A', H^*, x_1, x_2, y_1, y_2 are defined as in Lemma 4.0.9, and t_1, t_2, R_1, r_1 are defined after Lemma 4.0.3. And we say that H' is the main A'-B' core in γ.

We now study the structure of G inside the main A'-B' core H'.

Lemma 5.0.1 *$(H' - y_1, A'[x_1, x_2], y_2)$ is planar, the degree of y_2 in $H' - y_1$ is at least 2, and $H' - y_1 - A'(x_1, x_2)$ contains disjoint paths from y_1, y_2 to x_i, x_{3-i}, respectively, for $i \in [2]$. Moreover, for $i \in [2]$, let X_i be the path from x_i to y_2 on the outer walk of $H' - y_1$ without going through x_{3-i}, then $N(y_1) \cap V(H' - y_1 - A') \not\subseteq V(X_i)$ for $i \in [2]$.*

Proof. We can apply the same proof in Lemma 3.0.4, and show that $(H' - y_1, A'[x_1, x_2], y_2)$ is planar, and $N(y_1) \cap V(H' - y_1 - A') \not\subseteq V(X_i)$ for $i = 1, 2$.

Moreover, since $V(H - y_1) \subseteq V(H' - y_1)$, then, by (iii) of Lemma 4.0.1, the degree of y_2 in $H' - y_1$ is at least 2, and $H' - A'(x_1, x_2) - \{y_1x_1, y_1x_2\}$ contains disjoint paths from y_1, y_2 to x_1, x_2, respectively, as well as disjoint paths from y_1, y_2 to x_2, x_1, respectively. □

Lemma 5.0.2 *Let R be an A'-B' path from $r \in A'(x_1, x_2)$ to $r' \in B'[r_1, y_1)$ such that $B'[r_1, r']$ is minimal. If $r' \neq r_1$ then the following conclusions hold:*

(i) There exists an A-B core H_1 with r_1 as a foot.

(ii) Let r_2 be the other foot of H_1, then there exists an A'-B' bridge with r_1 as a foot, intersecting A' only at x_j for some $j \in [2]$, and lying on $B'[r_1, r_2]$.

(iii) $r' \in V(B'(r_1, r_2))$*, and* G *has an* A'*-*B' *bridge with feet* l'_1, r'_1*, which is internally disjoint from* R *and intersecting* A' *only at* x_j*, such that* $r' \in B'(l'_1, r'_1)$*.*

(iv) If G'_0 *has a cut* $\{a'_0, b'_1, b'_2\}$ *separating* $B'[b'_1, b'_2]$ *from* $\{a_0, b_1, b_2\}$ *such that* $b'_1 \in B'(r_1, r']$ *and* $b'_2 \in B'[y_2, b_2]$*, then* $r_1 = b_1$ *and* $a'_0 = a_0$*;* G'_0 *has no path from* a_0 *to* b_1 *and internally disjoint from* B'*, and* $\alpha(A', B') \leq 1$*.*

Proof. To prove (i), assume that r_1 is not a foot of any A-B core. Then by the definition of r_1, G has an edge from r_1 to $a' \in V(A(x_1, x_2))$. Since $r' \neq r_1$, $a' \notin A'(x_1, x_2)$. Moreover, a' is not contained in any A'-B' bridge lying on $B'[r_1, y_1]$, as any such A'-B' bridge is contained in an A-B' bridge (by (ii) of Lemma 4.0.9). So $a' \in V(H' - y_1) \setminus V(A')$. Hence, by (iv) of Lemma 4.0.9, $H' - y_1$ has a path Y_2 from a' to y_2 and internally disjoint from A'. Therefore, A' and $B'[b_1, r_1] \cup r_1a' \cup Y_2 \cup B'[y_2, b_2]$ are disjoint paths from a_1, b_1 to a_2, b_2, respectively, in $G - V(A'_0(B') - B') \cup \{y_1\}$, contradicting (i) of Lemma 3.0.2.

Now, we prove (ii). By Lemma 4.0.4, r_2 is the main foot of H_1. Hence, by (iii) of Lemma 4.0.1, r_1 has two neighbors u_1, u_2 in $H_1 - r_2 - A$. Since $B'[r_1, r_2]$ is induced in $G - \{r_1r_2\}$ (by Lemma 4.0.3), $u_p \notin B'$ for some $p \in [2]$. Moreover, $u_p \notin A'(x_1, x_2)$ since $r' \neq r_1$. Thus, u_p must be contained in some A'-B' bridge M_0 lying on $B'[r_1, r_2]$, which must have r_1 as a foot and cannot have both x_1 and x_2 as extreme hands (by (v) of Lemma 4.0.9). Hence, since $r' \neq r_1$, this A'-B' bridge intersect A' only at x_j for some $j \in [2]$.

Obviously, since G^* is 6-connected, $r' \in B'(r_1, r_2)$ to avoid the cut $\{r_1, r_2, x_1, x_2\}$ in G^* separating $V(H_1)$ from $\{a_0, a_1, a_2, b_1, b_2\}$. Let l'_0, r'_0 be the feet of M_0 with $l'_0 = r_1$ and $r'_0 \in B'[r_1, r_2]$. For, suppose (iii) fails. Then $r' \in B'[r'_0, r_2]$. Since $x_{3-j} \notin V(H_1 \cap A')$ (by Lemma 4.0.8), then by the definition of r', $\{x_j, r_1, r'\}$ is a cut in G separating M_0 from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

To prove (iv), we observe that $B'[r_1, r_2]$ is on the boundary of a finite face of G'_0. Therefore, since $r' \in B'(r_1, r_2)$, a'_0 and r_1 are also incident with that finite face. Suppose $r_1 \neq b_1$ or $a'_0 \neq a_0$. Then $\{a'_0, r_1, b'_2\}$ is a 3-cut in G'_0 separating $B'[r_1, b'_2]$ from $\{a_0, b_1, b_2\}$.

By Lemma 4.0.7, $r_1 = b_1$. So $a_0' \neq a_0$. Then, by Lemma 4.0.7, $\{a_0', b_1, b_2', b_2\}$ is a cut in G separating a_0 from $\{a_1, a_2\}$, a contradiction. So, $r_1 = b_1$ and $a_0' = a_0$. Hence, G_0' has no path that is from a_0 to b_1 and internally disjoint from B'. In particular, $\alpha(A', B') \leq 1$. □

Since G^* is 6-connected, G has two disjoint A'-B' paths from $p, q \in V(A'(x_1, x_2))$ to $p', q' \in V(B'[r_1, y_1))$, respectively. We choose P, Q to first maximize $A'[p, q]$, then minimize $B'[b_1, p'] \cap B'[b_1, q']$, and finally maximize $B'[p', q']$. By the symmetry between a_1 and a_2, we may relabel a_1, x_1, x_2, a_2 so that

- a_1, x_1, p, q, x_2, a_2 occur on A' in order, and $b_1, r_1, p', q', y_1, b_2$ occur on B' in order.

Lemma 5.0.3 *Any A'-B' path from $B'[r_1, p')$ to $A'(x_1, x_2)$ must be disjoint from P, Q, and end in $A'(p, q)$. Moreover, if $H' - y_1$ contains a path from $u \in A'[q, x_2)$ to y_2 and internally disjoint from A', then all A'-B' paths from $A'(u, x_2)$ to $B'[r_1, y_1]$ and internally disjoint from $H' - y_1$, are edges ending in $\{r', y_1\}$.*

Proof. First, assume S is an A'-B' path from $s' \in V(B'[r_1, p'))$ to $s \in V(A'(x_1, x_2))$. Then $V(S \cap (P \cup Q)) = \emptyset$; for otherwise, let $v \in V(S \cap (P \cup Q))$ with $S[s', v]$ minimal then $P' := S[s', v] \cup P[v, p]$ and Q (when $v \in V(P)$) or P and $Q' := S[s', v] \cup Q[v, q]$ (when $v \in V(Q)$) contradict the choice of P, Q. Hence, $s \in A'(p, q)$ as otherwise S, P or S, Q contradict the choice of P, Q.

Now let Y_2 be a path in $H' - y_1$ from $u \in V(A'[q, x_2))$ to y_2 and internally disjoint from A'. We first see that G has no path from $A'(u, x_2)$ to $B'[r_1, y_1) - p'$. For, suppose not. Let S be an A'-B' path from $s \in V(A'(u, x_2))$ to $s' \in V(B'[r_1, y_1) - p')$. Then $V(S \cap P) \neq \emptyset$, or else, P, S contradict the choice of P, Q. Since $s' \neq p'$, S, P are contained in an A'-B' bridge. However, by $u \in A'(p, s)$, the existence of Y_2 contradicts (v) of Lemma 4.0.9.

Now let S be an arbitrary A'-B' path from $s \in A'(u, x_2)$ to $s' \in B'[r_1, y_1]$. Suppose S has length at least 2. Then S is contained in some A'-B' bridge N with feet n_1', n_2' and extreme hands n_1, n_2. Then $n_1', n_2' \in \{p', y_1\}$. By (v) of Lemma 4.0.9 and the existence of S and Y_2, $A'[n_1, n_2] \subseteq A[u, x_2]$. Let $h_1, h_2 \in A'[x_1, x_2]$, such that $A'[n_1, n_2] \subseteq A'[h_1, h_2]$,

$H' - y_1$ does not contain a path from $A'(h_1, h_2)$ to y_2 and internally disjoint from A', and subject to this, $A'[h_1, h_2]$ is maximal. Clearly, $A'(h_1, h_2) \subseteq A'(u, x_2)$, and for $i \in [2]$, $H' - y_1$ contains a path from h_i to y_2 and internally disjoint from A'. By (v) of Lemma 4.0.9, $\{h_1, h_2, p', y_1\}$ is a cut in G^* separating $V(N)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Thus, S must be an edge. To complete the proof, we need to show $r' = p'$. For, suppose $r' \neq p'$. By (i), R is disjoint from P, Q with $r \in A'(p, q)$, and so R, P, S, Y_2 force a double cross in A, B, contradicting Lemma 2.0.7. □

Let $R = P$ if $r' = p'$, and if $r' \neq p'$ then by Lemma 5.0.3, R is disjoint from P, Q with $r \in A'(p, q)$. By Lemma 5.0.1, for $i \in [2]$, we let $P_{1,i}, P_{2,3-i}$ be disjoint paths from x_1, x_2 to y_i, y_{3-i}, respectively, in $H' - y_1 - A'(x_1, x_2)$.

We now use the structure inside H' to derive further structure outside H'.

Lemma 5.0.4 *(i) G has no edge from $A'(x_2, a_2]$ to $B'(b_1, r_1]$ and no edge from $A'[a_1, x_1)$ to $B'[y_2, b_2)$.*

(ii) G has no edge from b_1 to $A'[a_1, x_1] \cup A'[x_2, a_2]$ and no edge from b_2 to $A'[x_2, a_2]$.

(iii) $r_1 = b_1$ implies $x_1 = a_1$, and $y_2 = b_2$ implies $x_2 = a_2$.

(iv) If $y_2 \neq b_2$ and y_2 is a cut vertex of G'_0 separating b_2 from $\{a_0, b_1\}$, then $N(b_2) = \{y_2, x_1\}$, $a_1 \neq x_1$, and $a_2 = x_2$.

Proof. By Lemma 4.0.7 and (iv) of Lemma 5.0.2, we may assume

(1) when $b_1 \neq r_1$, $G'_0 - B'(b_1, r'] - B'[y_2, b_2]$ contains disjoint paths B_1^*, A_0^* from b_1, a_0 to q', y_1, respectively.

(2) G has no edge from $A'(x_2, a_2]$ to $B'(b_1, r_1]$.

For, let $e = ab \in E(G)$ with $a \in A'(x_2, a_2]$ and $b \in B'(b_1, r_1]$. Then $b_1 \neq r_1$; so B_1^*, A_0^* exist by (1). Now $A'[a_1, r] \cup R \cup B'[b, r'] \cup e \cup A'[a, a_2] \cup P_{1,1} \cup A_0^*$ and $B_1^* \cup Q \cup A'[q, x_2] \cup P_{2,2} \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction. □

(3) G has no edge from b_2 to $A'[x_2, a_2]$.

For, let $e = ab_2 \in E(G)$ with $a \in A'[x_2, a_2]$. Then $a \neq a_2$ and let $e' = a_2b' \in E(G)$ with $b' \in B'(b_1, b_2)$. Now $b' \notin B'[y_2, b_2)$ to avoid the double cross $e, e', P_{1,2}, P_{2,1}$. Hence, $b' \in B'(b_1, r_1]$, contradicting (2). □

(4) G has no edge from $A'[a_1, x_1)$ to $B'[y_2, b_2)$.

Otherwise, let $e = ab \in E(G)$ with $a \in A'[a_1, x_1)$ and $b \in B'[y_2, b_2)$. Then G has no edge from b_2 to $\{x_1, x_2\}$; as such an edge must be b_2x_1 by (3), which forms a double cross with e, $P_{1,1}$ and $P_{2,2}$, contradicting Lemma 2.0.7.

Hence, by Lemma 4.0.7 and (iv) of Lemma 5.0.2, $G_0' - B'[b_1, r'] - B'[y_2, b]$ has disjoint paths B_2, A_0 from b_2, a_0 to y_1, q', respectively. But then, $A'[a_1, a] \cup e \cup B'[y_2, b] \cup P_{2,2} \cup A'[q, a_2] \cup Q \cup A_0$ and $B'[b_1, r'] \cup R \cup A'[x_1, r] \cup P_{1,1} \cup B_2$ show that γ is feasible, a contradiction. □

(5) (i)–(ii) hold.

For, suppose not. Then G has an edge $e = b_1a$ with $a \in A'[a_1, x_1] \cup A'[x_2, a_2]$.

Suppose $a \in A'[a_1, x_1]$. Then $a \neq a_1$, and let $e' = a_1b' \in E(G)$ with $b' \in B'(b_1, b_2)$. Now $b' \notin B'(b_1, r_1]$ to avoid the double cross $e, e', P_{1,2}, P_{2,1}$. So $b' \in B'[y_2, b_2)$, contradicting (4).

Hence, $a \in A'[x_2, a_2]$. Then $a \neq a_2$, and let $e' = a_2b' \in E(G)$ with $b' \in B'(b_1, b_2)$. Now $b' \notin B'(b_1, r_1]$ to avoid the double cross $e, e', P_{1,1}, P_{2,2}$. Hence, $b' \in B'[y_2, b_2)$.

If G has an edge e_3 from b_2 to $\{x_1, x_2\}$ then, by (3), it ends with x_1. So $a_1 \neq x_1$, and G has an edge e_4 from a_1 to $B'(b_1, b_2)$. But now, e, e', e_3, e_4 force a double cross, a contradiction.

So G has no edge from b_2 to $\{x_1, x_2\}$. Hence, by Lemma 4.0.7, $G_0' - B'[b_1, r_1] - B'[y_2, b']$ has disjoint paths B_2, A_0 from b_2, a_0 to y_1, q', respectively. But then, $A'[a_1, q] \cup$

$P_{1,2} \cup B'[y_2, b'] \cup e' \cup Q \cup A_0$ and $e \cup A'[x_2, a] \cup P_{2,1} \cup B_2$ show that γ is feasible, a contradiction. $\Box$

Since G^* is 6-connected, it follows from (2) and (4) that (iii) holds. It remains to prove (iv). So assume $y_2 \neq b_2$ and y_2 is a cut vertex of G'_0 separating b_2 form $\{a_0, b_1\}$. Then $\alpha(A', B') \leq 1$.

Suppose $B'(y_2, b_2) \neq \emptyset$. Then, since G^* is 6-connected, it follows from (4) that G has edges from $B'(y_2, b_2)$ to distinct $u_1, u_2 \in V(A'[x_2, a_2])$, and we choose u_1, u_2 so that $A'[u_1, u_2]$ is maximal. Now, by (2) and (3), $\{u_1, u_2, y_2, b_2, x_1\}$ is a cut in G^* separating $V(B'(y_2, b_2))$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

So $B'(y_2, b_2) = \emptyset$. Then $a_2 = x_2$; for otherwise, since G^* is 6-connected, G has an edge from a_2 to $B'(b_1, r_1]$, contradicting (2). We may assume that there exists $e = b_2a \in E(G)$ with $a \in A'(a_1, x_1)$; as otherwise, (iv) holds. Let $e' = a_1b' \in E(G)$ with $b' \in B'(b_1, b_2)$. Then $b' \in B'(b_1, r_1]$ by (4); so $b_1 \neq r_1$, and B_1^*, A_0^* exist by (1). Now, by Lemma 3.0.1, we derive $\alpha(A', B') = 2$ with the following paths: the path $e' \cup B'[b_1, b']$ from a_1 to b_1, the path $A'[a_1, a] \cup e$ from a_1 to b_2, the path $B_1^* \cup Q \cup A'[x_1, q] \cup P_{1,2} \cup B'[y_2, b_2]$ from b_1 to b_2, and the path $A_0^* \cup P_{2,1}$ from a_0 to a_2. This contradicts $\alpha(A', B') \leq 1$ as A', B' is a good frame. $\Box$

Let H_0 denote the minimal union of blocks of $H' - y_1 - A'[q, x_2]$ containing X_1, let W denote the path between x_1 and y_2, such that W is contained in the outer walk of H_0, and for any vertex $v \in V(W - A')$, there exists a vertex $u \in A'[q, x_2]$, such that u, v are incident with a finite face of $H' - y_1$, and let $w_1 \in V(A' \cap W)$ with $A'[x_1, w_1]$ maximal.

Next, we further study the structure inside H'.

Lemma 5.0.5 *(i) $H_0 = H' - y_1 - A(w_1, x_2]$, and each vertex in $W(w_1, y_2]$ has at most two neighbors on $A'[q, x_2]$, inducing a subpath of A' with vertices at most two.*

(ii) $H' - \{y_1, y_2\} - A'(x_1, x_2)$ contains a path from x_1 to x_2.

Proof. Suppose (i) is not true. Then $H' - y_1$ has a nontrivial $(H_0 \cup A'[q, x_2])$-bridge J which has exactly one vertex in $W(w_1, y_2]$ (by definition of H_0 and since $G - A'$ is connected) or some vertex $w \in V(W(w_1, y_2])$ has two neighbors on $A'[q, x_2]$ such that the subpath of A' between them has at least three vertices. In the first case, let $w \in V(J \cap H_0)$ and $u, v \in V(J \cap A')$ such that $J \cap A' \subseteq A'[u, v]$; and in the second case, let u, v be the neighbors of w on $A'[q, x_2]$ such that $A'[u, v]$ is maximal. Then by Lemma 5.0.3, $\{u, v, w, y_1, r'\}$ is a cut in G^*, a contradiction.

Now suppose (ii) is not true. Then there exists $v_0 \in V(A'(x_1, x_2))$ such that y_2, v_0 are incident with a finite face of $H' - y_1$. We further choose v_0 so that $A'[v_0, x_2]$ is minimal, and let (L_1, L_2) be a separation in $H' - y_1$ such that $V(L_1 \cap L_2) = \{y_2, v_0\}$, $x_1 \in V(L_1)$, and $x_2 \in V(L_2)$.

By Lemma 5.0.1, for each $j \in [2]$, $H' - A'(x_1, x_2)$ contains disjoint paths from y_1, y_2 to x_j, x_{3-j}, respectively. So for $j \in [2]$, $G[V(L_j) \cup \{y_1\}] - y_2$ contains a path T_j from y_1 to x_j and internally disjoint from A'.

We see that y_2, v_0 are not incident with a common finite face of H_0. For otherwise, $v_0 \in A'(x_1, w_1]$, $x_1 \neq w_1$, and $W[w_1, y_2] \subseteq L_2$. Hence, T_1, $W[w_1, y_2]$, P and Q are disjoint, which form a doublecross, a contradiction to Lemma 2.0.7.

Now, by the minimality of $A'[v_0, x_2]$ and planarity of $H' - y_1$, $v_0 \in A'[q, x_2)$. Therefore, by Lemma 5.0.3, $\{v_0, x_2, r', y_1, y_2\}$ is a cut in G^* separating $V(L_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. $\square$

Let $w_2, \cdots, w_m$ be the vertices on W in order from x_1 to y_2 such that for $i \in \{2, \cdots, m\}$, w_i has a neighbor on $A'[q, x_2]$, and for $i \in \{2, \cdots, m\}$, let $u_i, v_i \in N(w_i \cap A')$, such that a_1, u_i, v_i, a_2 occur on A' in order with $A'[u_i, v_i]$ maximal.

Lemma 5.0.6 *$w_1 \neq x_1$, and H_0 is 2-connected.*

Proof. Suppose this is false. Let $z \in V(H_0)$ such that $z = x_1$ (when $x_1 = w_1$) or z is a cut vertex of H_0 and, subject to this, $W[x_1, z]$ is maximal. Then $V(W[z, y_2] \cap X_1) = \{z, y_2\}$. Note that $z \in X_1[x_1, y_2)$ and $w_m \in W(z, y_2)$.

Let k be minimum such that $w_k \in W(z, y_2]$ and $u \in N(w_k) \cap V(A'[q, x_2])$ such that $A'[q, u]$ is minimal. Moreover, let K denote the $\{z, u\}$-bridge of $H' - y_1$ containing $A'[u, x_2] \cup X_2$, and let $K^* := G[V(K) \cup \{y_1\}]$.

By (v) of Lemma 4.0.9 and by the existence of $W[y_2, w_k] \cup w_k u$,

(1) no A'-B' bridge has one extreme hand in $A'[x_1, u)$ and the other in $A'(u, x_2]$.

Thus, since $\{y_1, y_2, z, u, x_2\}$ is not a cut in G^* separating $V(K)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, G has an A'-B' path from $A'(u, x_2)$ to $B'[r_1, y_1)$ and internally disjoint from H'. By Lemma 5.0.3,

(2) all A'-B' paths from $A'(u, x_2)$ to $B'[r_1, y_1]$ and internally disjoint from H' are edges from $A'(u, x_2)$ to $\{r', y_1\}$.

So let $e = ar' \in E(G)$ with $a \in A'[u, x_2)$, and choose a such that $A'[u, a]$ is minimal.

(3) Let L denote the path on the outer walk of K between y_2 and u not going through x_2, and let $L_0 := L \cup A'[u, a]$. Then $V(L_0 \cap X_2) = \{y_2\}$, and $N(y_1) \cap V(K) \subseteq V(L_0)$.

First, suppose there exists $v \in V(L_0 \cap X_2)$, such that $v \neq y_2$. Then $\{v, y_1, u, x_2, r'\}$ is a cut in G^* separating $V(A'(u, x_2))$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Now suppose there exist $v \in N(y_1) \cap V(K)$ such that $v \notin V(L_0)$. We claim that $K^* - V(L_0)$ has a path Y_1 from y_1 to x_2. For otherwise, by the planar structure of K, there exist $c_1, c_2 \in V(L_0)$, such that c_1, c_2 are incident with a finite face of K, and $\{c_1, c_2\}$ is a 2-cut in K separating v from x_2. Thus, by (2) and the choice of a, $\{c_1, c_2, y_1, u, z\}$ is a cut in G^* separating v from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

If G has an A'-B' path T from $A'(x_1, u)$ to $B'(r', y_1]$ and internally disjoint from H', then T, e, L, Y_1 force a double cross, a contradiction. So T does not exist. Then $u = q$ and, by (1), $\{x_1, u, z, r'\}$ is a cut in G^* separating r from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

We will need the following claim.

(4) G_0' contains a path A_0^* from $B'(r', y_1)$ to a_0 and internally disjoint from B'.

For otherwise, there exists $b_1' \in V(B'[b_1, r'])$, such that $\{b_1', y_1\}$ is a 2-cut in G_0' separating $B'[b_1', y_1]$ from $\{a_0, b_1, b_2\}$. Furthermore, $\{b_1', y_1, y_2\}$ is a 3-cut in G_0' separating $B'[b_1', y_2]$ from $\{a_0, b_1, b_2\}$. We choose b_1' so that $B'[b_1, b_1']$ is minimal. By Lemma 4.0.7 and (iv) of Lemma 5.0.2, $b_1' = b_1$, and $\{b_1, y_1, y_2, b_2\}$ is a cut in G^* separating a_0 from $\{a_1, a_2\}$, a contradiction. □

Let $y_1', y_1'' \in V(L_0) \cap N(y_1)$ such that a, y_1', y_1'', y_2 occur on L_0 in order and, subject to this, $L_0[y_1', y_1'']$ is maximal.

(5) $y_1'' \in L_0[z, u)$.

For, otherwise, $y_1'' \in L_0(z, y_2]$. Then $y_1' \notin L_0[z, y_2]$; otherwise, G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{r', u, z, y_1, y_2, x_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $G_2 = K^*$, and $(G_2, r', u, z, y_1, y_2, x_2)$ is planar, which contradicts Lemma 2.0.3.

We claim that $K - V(L_0[y_1', a] \cup L_0[y_2, y_1''])$ contains a path X' from x_2 to z. For otherwise, by (3) and the planar structure of K, there exist $c_1 \in V(L_0[y_1', a])$ and $c_2 \in V(L_0[y_2, y_1''])$, such that c_1, c_2 are incident with a finite face of K, and $\{c_1, c_2\}$ is a 2-cut in K separating x_2 from z. If $c_1 \in A'[u, a]$ then $\{c_1, c_2, y_2, x_2, r'\}$ is a cut in G^* separating $V(X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. So $c_1 \notin A'[u, a]$. Then G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{r', u, c_1, c_2, y_2, x_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(A'[u, x_2] \cup X_2) \subseteq V(G_2)$, and $(G_2, r', u, c_1, c_2, y_2, x_2)$ is planar. This contradicts Lemma 2.0.3.

Now, the following paths give a contradiction to (i) of Lemma 3.0.2: the path $A'[a_1, x_1] \cup X_1[x_1, z] \cup X' \cup A'[x_2, a_2]$ from a_1 to a_2, the path $B'[b_1, r'] \cup e \cup L_0[a, y_1'] \cup y_1'y_1 \cup y_1y_1'' \cup L_0[y_1'', y_2] \cup B'[y_2, b_2]$ from b_1 to b_2, and the path A_0^* from $B'(r', y_1)$ to a_0. □

We claim that $y_1' \in A'(u, a]$. For, otherwise, $y_1', y_2'' \in L_0[z, u]$. Now, G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{r', u, y_1, z, y_2, x_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $G_2 = K^*$, and $(G_2, r', u, y_1, z, y_2, x_2)$ is planar. This contradicts Lemma 2.0.3.

Moreover, $K - V(L_0[y_1', a] \cup L_0[y_2, y_1''])$ contains a path X' from x_2 to u. For otherwise, by (3) and the planar structure of K, there exist $c_1 \in V(L_0[y_1', a])$ and $c_2 \in V(L_0[y_2, y_1''])$, such that c_1, c_2 are incident with a finite face of K, and $\{c_1, c_2\}$ is a 2-cut in K separating x_2 from u. If $c_2 \in L_0[y_2, z]$ then $\{c_1, c_2, y_2, x_2, r'\}$ is a cut in G^* separating $V(X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. So $c_2 \notin L_0[y_2, z]$. Then G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{r', c_1, c_2, z, y_2, x_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(A'[c_1, x_2] \cup X_2) \subseteq V(G_2)$, and $(G_2, r', c_1, c_2, z, y_2, x_2)$ is planar. This contradicts Lemma 2.0.3.

Hence, the following paths contradict (i) of Lemma 3.0.2: the path $A'[a_1, u] \cup X' \cup A'[x_2, a_2]$ from a_1 to a_2, the path $B'[b_1, r'] \cup e \cup L_0[a, y_1'] \cup y_1'y_1 \cup y_1y_1'' \cup L_0[y_1'', y_2] \cup B'[y_2, b_2]$ from b_1 to b_2, and the path A_0^* from $B'(r', y_1)$ to a_0. □

Lemma 5.0.7 *Let $z_1, z_2 \in V(W)$ with $W[z_1, z_2]$ is maximal, such that x_1, z_1, z_2, y_2 occur on W in order, and for each $i \in [2]$, $G[H_0 + y_1]$ has a path Z_i from y_1 to z_i and internally disjoint from W. Then, $N(y_1) \cap V(X_1[x_1, y_2)) = \emptyset$ and $Z_1 \cap (X_1 \cup X_2) = \emptyset$.*

Proof. By Lemma 5.0.6, $w_1 \neq x_1$ and H_0 is 2-connected. So $V(X_1 \cap W) = \{x_1, y_2\}$.

If $N(y_1) \cap V(X_1[x_1, y_2)) \neq \emptyset$ or $Z_1 \cap X_1 \neq \emptyset$ then $Z_1 \cup X_1$ contains a path S from y_1 to x_1 and disjoint from $W[w_1, y_2]$. Now S, $W[w_1, y_2]$, P, and Q force a double cross, contradicting Lemma 2.0.7. So $N(y_1) \cap V(X_1[x_1, y_2)) = \emptyset$ and $Z_1 \cap X_1 = \emptyset$.

Moreover, $Z_1 \cap X_2 = \emptyset$. For, otherwise, by the choice of z_1 and Z_1, it follows from the planarity of $H' - y_1$ that $z_1 \in V(X_2)$. But then, $H' - A'(x_1, x_2)$ contains no disjoint paths from y_1, y_2 to x_1, x_2, respectively. This contradicts Lemma 5.0.1. □

Lemma 5.0.8 *$a_2 = x_2$, and if $y_2 \neq b_2$ then y_1, y_2 are cut vertices in G_0' separating b_2 from $\{a_0, b_1\}$, $N(b_2) = \{y_2, x_1\}$, and $a_1 \neq x_1$. Moreover, one of the following holds:*

(i) there exists a 2-cut $\{z_1', z_2'\}$ in H_0 with $x_1, z_1', z_1, z_2, z_2', y_2$ on W in order such that $W(z_1', z_2') \neq \emptyset$ and z_1', z_2' are incident with a finite face of H_0, or

(ii) $N(y_1) \cap V(H_0) \subseteq V(W[w_1, y_2])$ and, for any $i \in [m]$, $w_i \notin W(z_1, z_2)$.

Proof. By Lemma 5.0.6, $w_1 \neq x_1$, and H_0 is 2-connected. If $y_2 = b_2$, then by (iii) of Lemma 5.0.4, we have $a_2 = x_2$.

Now assume $y_2 \neq b_2$. We claim that G'_0 has a 3-cut $\{a'_0, b'_1, y_2\}$ with $b'_1 \in B'[b_1, r_1]$, which separates $B'[b'_1, y_2]$ from $\{a_0, b_1, b_2\}$. For otherwise, by (iv) of Lemma 5.0.2, $G'_0 - B'[b_1, r'] - y_2$ contains disjoint paths A_0, B_2 from a_0, b_2 to q', y_1, respectively. Let Y_1 be a path in $Z_1 \cup W[z_1, w_1] \cup A'[w_1, r]$ from y_1 to r. Note that $r \notin A'[q, x_2]$ and, by Lemma 5.0.7, $Y_1 \cap (A'[q, x_2] \cup X_1 \cup X_2) = \emptyset$. Now, $A'[a_1, x_1] \cup X_1 \cup X_2 \cup A'[q, a_2] \cup Q \cup A_0$ and $B'[b_1, r'] \cup R \cup Y_1 \cup B_2$ show that γ is feasible, a contradiction.

Thus, when $y_2 \neq b_2$, we may apply Lemma 4.0.7 (with $b'_2 = y_2$), and conclude that $b'_1 = b_1$, $a'_0 = a_0$, and y_1, y_2 are cut vertices in G'_0 separating b_2 from $\{a_0, b_1\}$. By (iv) of Lemma 5.0.4, we have $N_G(b_2) = \{y_2, x_1\}$, $a_1 \neq x_1$, and $a_2 = x_2$.

We now show (i) or (ii) holds. First, suppose $z_1 = z_2$. Then $N(y_1) \cap V(H_0) = \{z_1\}$; or else, there exists $v \in N(y_1) \cap V(H_0)$ with $v \neq z_1$, and $\{z_1, y_1\}$ is a cut in G separating v from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. Clearly, $z_1 \in V(W(w_1, y_2))$, and so (ii) holds.

So we may assume $z_1 \neq z_2$. Now suppose $W(z_1, z_2) \cap \{w_1, \dots, w_m\} = \emptyset$. Then (ii) holds or there exists $v \in N(y_1) \cap V(H_0)$ such that $v \notin V(W)$. In the latter case, there exist $c_1, c_2 \in V(W(x_1, y_2])$, such that $\{c_1, c_2\}$ is a 2-cut in H_0 separating v from x_1; since, otherwise, $H_0 - W(x_1, y_2]$ contains a path T from v to x_1, and $y_1 v \cup T, W[w_1, y_2], R, Q$ force a double cross, contradicting Lemma 2.0.7. Now, $\{y_1, c_1, c_2\}$ is a cut in G^*, a contradiction.

Hence, we may assume $W(z_1, z_2) \cap \{w_1, \dots, w_m\} \neq \emptyset$. Now suppose (i) fails. Then by the planar structure of H_0, $H_0 - W(x_1, z_1] - W[z_2, y_2]$ contains a path X' from x_1 to $W(z_1, z_2)$ and internally disjoint from W.

We claim that X' must be disjoint from Z_1, Z_2. For otherwise, let $x^* \in V(X' \cap Z_j)$ for some $j \in [2]$. As X', Z_1, Z_2 are all internally disjoint from W, $Z_j[s_j, x^*] \cup X'[x^*, x_1]$ implies that $z_1 = x_1$, contradicting Lemma 5.0.7 that $V(Z_1 \cap (X_1 \cup X_2)) = \emptyset$.

We claim $w_1 \in W(z_1, z_2)$. For otherwise, $w_i \in W(z_1, z_2)$ for some $i \geq 2$. By Lemma 4.0.7 and (iv) of Lemma 5.0.2, there exists a path A^*_0 in G'_0 from a_0 to $B'(r', y_1)$ internally disjoint from B'. Now $A'[a_1, x_1] \cup X' \cup W(z_1, z_2) \cup w_i v_i \cup A'[q, a_2] \cup Q \cup B'(r', y_1) \cup A_0^*$ and $B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, z_1] \cup Z_1 \cup Z_2 \cup W[z_2, y_2] \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction.

So $z_1 \in A'(x_1, w_1)$. Moreover, $r \notin A'(x_1, z_1]$; otherwise, $A'[a_1, x_1] \cup X' \cup W(z_1, z_2) \cup A'[w_1, a_2] \cup Q \cup B'(r', y_1) \cup A_0^*$ and $B'[b_1, r'] \cup R \cup A'[r, z_1] \cup Z_1 \cup Z_2 \cup W[z_2, y_2] \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction. But now, $A'[a_1, z_1] \cup Z_1 \cup B'(r', y_1] \cup A_0^* \cup Q \cup A'[q, a_2]$ and $B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, y_2] \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction. □

Lemma 5.0.9 *Suppose (i) of Lemma 5.0.8 holds, and a 2-cut $\{z_1', z_2'\}$ in G_0' is chosen with $W[z_1', z_2']$ maximal. Then $z_1' \in A'[x_1, w_1]$.*

Proof. For, suppose 2-cut $\{z_1', z_2'\}$ is chosen with $W[z_1', z_2']$ maximal, and $z_1' \notin A'[x_1, w_1]$.

By Lemma 5.0.5, we can define u', u'', v', v'', such that $u', u'' \in V(A'[q, x_2])$, x_1, u', u'', x_2 occur on A' in order, $H' - y_1$ has edges from u', u'' to $v', v'' \in V(W(z_1', z_2'))$, respectively, subject to this, $A'[u', u'']$ is maximal, and subject to this, $W[v', v'']$ is maximal. Obviously, there exists a separation (K, K_0) in $H' - y_1$, such that $V(K \cap K_0) = \{u', u'', z_1', z_2'\}$, $V(W[z_1', z_2'] \cup A'[u', u'']) \subseteq V(K)$, and $V(W[x_1, z_1'] \cup X_1) \subseteq V(K_0)$. We also let $K^* := G[V(K) \cup \{y_1\}]$.

By (v) of Lemma 4.0.9 and by the existence of the paths from u', u'' to y_2, respectively, in $H' - y_1$,

(1) there does not exist an A'-B' bridge with extreme hands n_1, n_2, such that for some $v \in \{u', u''\}$, $n_1 \in A'[x_1, v)$ and $n_2 \in A'(v, x_2]$.

Now, since $\{y_1, z_1', z_2', u', u''\}$ is not a cut in G separating $V(K)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, then, combined with (1), we may assume

(2) $A'(u', u'') \neq \emptyset$, and G has an A'-B' path from $A'(u', u'')$ to $B'[r_1, y_1)$, internally disjoint from $H' - y_1$.

By Lemma 5.0.3, we may further assume

(3) all A'-B' paths from $A'(u', u'')$ to $B'[r_1, y_1]$, internally disjoint from $H' - y_1$, are edges from $A'(u', u'')$ to $\{r', y_1\}$.

Now, we let $e = ar' \in E(G)$ with $a \in A'[u', u'')$, such that $A'[u', a]$ is minimal.

(4) G'_0 contains a path A^*_0 from $B'(r', y_1)$ to a_0, internally disjoint from B'.

For otherwise, there exists $b'_1 \in B'[b_1, r']$, such that $\{b'_1, y_1\}$ is a 2-cut in G'_0 separating $B'[b'_1, y_1]$ from $\{a_0, b_1, b_2\}$. Furthermore, $\{b'_1, y_1, y_2\}$ is a 3-cut in G'_0 separating $B'[b'_1, y_2]$ from $\{a_0, b_1, b_2\}$. By Lemma 4.0.7 and (iv) of Lemma 5.0.2, $b'_1 = b_1$, and $\{b_1, y_1, y_2, b_2\}$ is a cut in G separating a_0 from $\{a_1, a_2\}$, a contradiction. □

Since z'_1 may not be chosen so that $z'_1 \in A'[x_1, w_1]$, then there does not exist a vertex $v \in A'[x_1, w_1]$, such that v, z'_2 are incident with a common finite face of K_0. Thus, we may assume

(5) $K_0 - V(A'[x_1, u'])$ contains a path Y from y_2 to z'_1, internally disjoint from A'.

Let L denote the path on the outer walk of K from z'_1 to u' without going through u''. Obviously, $z'_2 \notin V(L)$. Moreover, we let $L_0 := L \cup A'[u', a]$.

(6) $N(y_1) \cap V(K) \not\subseteq (V(L_0) \cup \{z'_2\})$.

For, suppose $N(y_1) \cap V(K) \subseteq (V(L_0) \cup \{z'_2\})$. Obviously, $V(L_0) \cap N(y_1) \neq \emptyset$; otherwise, $\{u', u'', z'_1, z'_2, r'\}$ is a cut in G separating $V(K)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Now, we let $y'_1, y''_1 \in V(L_0) \cap N(y_1)$, such that a, y'_1, y''_1, z'_1 occur on L_0 in order, and $L_0[y'_1, y''_1]$ is maximal.

We first claim $y'_1 \in L_0(u', a]$. For otherwise, $y'_1, y''_2 \in V(L_0[z'_1, u'])$. Now, G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{r', u', y_1, z'_1, z'_2, u''\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(K) \subseteq V(G_2)$, and $(G_2, r', u', y_1, z'_1, z'_2, u'')$ is planar, which contradicts Lemma 2.0.3.

Now, we see that $y_1'' \in L_0[z_1', u')$. For, suppose $y_1'' \notin L_0[z_1', u')$. Then $y_1'' \in L_0[u', a]$ and $z_2' \in N(y_1)$. Now, G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{r', y_1, u', z_1', z_2', u''\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(K) \subseteq V(G_2)$, and $(G_2, r', y_1, u', z_1', z_2', u'')$ is planar, which contradicts Lemma 2.0.3.

Then we claim that $K - V(L_0[z_1', y_1''] \cup L_0[y_1', a]) \cup \{z_2'\}$ contains a path X' from u'' to u'. For otherwise, by the planar structure of K, there exist $c_1 \in V(L_0[y_1', a])$, $c_2 \in V(L_0[z_1', y_1'']) \cup \{z_2'\}$, such that c_1, c_2 are incident with a common finite face of K, and $\{c_1, c_2\}$ is a 2-cut in K separating u' from u''. By the existence of the path $u''v'' \cup W[v'', v'] \cup v'u'$ from u'' to u', we may assume $c_2 = v'$. Moreover, $v' \neq v''$; otherwise, $\{v', u', u'', r', y_1\}$ is a cut in G separating $V(A'(u', u''))$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. But then, as G^* is 6-connected, G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{r', c_1, v', z_1', z_2', u''\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(A'[c_1, u'']) \cup \{v''\} \subseteq V(G_2)$, and $(G_2, r', c_1, v', z_1', z_2', u'')$ is planar, which contradicts Lemma 2.0.3.

Now, the path $A'[a_1, u'] \cup X' \cup A'[u'', a_2]$ from a_1 to a_2, the path $B'[b_1, r'] \cup e \cup L_0[a, y_1'] \cup y_1'y_1 \cup y_1y_1'' \cup L_0[y_1'', z_1'] \cup Y \cup B'[y_2, b_2]$ from b_1 to b_2, and the path A_0^* from $B'(r', y_1)$ to a_0 contradict (i) of Lemma 3.0.2. □

(7) $K^* - V(L_0) \cup \{z_2'\}$ contains a path Y_1 from y_1 to u''.

For, suppose (7) fails. By (6), there exists $v \in N(y_1) \cap V(K)$, such that $v \notin V(L_0) \cup \{z_2'\}$.

Since $K^* - V(L_0) \cup \{z_2'\}$ contains no path from y_1 to u'', then by the planar structure of K, there exist $c_1, c_2 \in V(L_0) \cup \{z_2''\}$, such that c_1, c_2 are incident with a common finite face of K, and $\{c_1, c_2\}$ is a 2-cut in K separating v from u''. Thus, combined with (3) and the choice of a, $\{c_1, c_2, y_1, u', z_1'\}$ is a cut in G separating v from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

(8) G has no A'-B' path from $A'[a_1, u')$ to $B'(r', y_1]$, internally disjoint from $H' - y_1$.

For, suppose G has an A'-B' path T from $A'[a_1, u')$ to $B'(r', y_1]$, internally disjoint from $H' - y_1$. Then $T, e, Y \cup L, Y_1$ force a doublecross, a contradiction. □

(9) $b_1 = r_1 = r'$.

We may assume $b_1 = r_1$ and so $a_1 = x_1$ by (iii) of Lemma 5.0.4. For, suppose $b_1 \neq r_1$. By Lemma 4.0.7 and (iv) of Lemma 5.0.2, $G'_0 - r' - B'[y_2, b_2]$ contains disjoint paths B_1, A_0 from b_1, a_0 to q', y_1, respectively. Now, $A'[a_1, r] \cup R \cup e \cup A'[a, a_2] \cup Y_1 \cup A_0$ and $B_1 \cup Q \cup A'[q, u'] \cup L \cup Y \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction.

We may assume $r_1 = r'$. For, suppose $r_1 \neq r'$. By (iii) of Lemma 5.0.2, there exists an A'-B' bridge M_4 with feet l'_4, r'_4, such that R is internally disjoint from M_4, and $r' \in B'(l'_4, r'_4)$. Let P^* be the path from l'_4 to r'_4 in M_4, internally disjoint from A', B', and let A'_0 be the path from a_0 to y_1 in G'_0, internally disjoint from B', then $A'[a_1, r] \cup R \cup e \cup A'[a, a_2] \cup Y_1 \cup A'_0$ and $B'[b_1, l'_4] \cup P^* \cup B'[r'_4, q'] \cup Q \cup A'[q, u'] \cup L \cup Y \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction. □

Now, by (8), (9), Lemma 5.0.8, and Lemma 5.0.3, $\{b_1, u', a_2, y_1, b_2\}$ is a cut in G separating a_0 from a_1, a contradiction. □

Lemma 5.0.10 *(i) $\alpha(A', B') = 1$, and y_1 is a cut vertex in G'_0 separating b_2 from $\{a_0, b_1\}$;*

(ii) Let A'_0 be the path from y_1 to a_0 in G'_0, internally disjoint from B' and on the boundary of G'_0, then $G'_0 - B'(b_1, r'] - A'_0$ contains a path B'_1 from b_1 to q'.

Proof. We may assume $H' - \{y_1, y_2\} - Z_1 \cup W[z_1, w_1] \cup A'(x_1, x_2)$ contains a path X_0 from x_1 to x_2. For otherwise, by the planar structure of $H' - y_1$, there exists a vertex $v \in V(Z_1 \cup W[z_1, w_1] \cup A'(x_1, x_2))$, such that y_2, v are incident with a common finite face of H_0. By Lemma 5.0.5, $v \notin A'(x_1, x_2)$, and so $v \in V(Z_1 \cup W[z_1, w_1])$. Now, we claim that there exists $c \in W[x_1, z_1]$, such that $\{y_2, c\}$ is a cut in H_0 separating $W(c, y_2)$ from x_1. For otherwise, $v \notin W[z_1, w_1]$. So $v \in Z_1[s_1, z_1)$, and (i) of Lemma 5.0.8 holds with 2-cut $\{z'_1, z'_2\}$. But then, z'_1, v, y_2 are incident with a common finite face of H_0, and $\{y_2, z'_1\}$ is a 2-cut, which still leads to our claim. Thus, our claim is true. Now, the existence of

$\{y_2, c\}$ show that (i) of Lemma 5.0.8 holds. By Lemma 5.0.9, $\{y_2, c\}$ may be chosen with $c \in A'[x_1, w_1]$, a contradiction to Lemma 5.0.5.

We now prove y_1 is a cut vertex in G'_0 separating b_2 from $\{a_0, b_1\}$. For, suppose not. Then by Lemma 5.0.8, we may assume $y_2 = b_2$.

Moreover, we may assume $G'_0 - B'[b_1, r'] - B'(y_1, b_2)$ contains disjoint paths A_0, B_2 from a_0, b_2 to q', y_1, respectively. For otherwise, by planar structure of G'_0, there exists a 3-cut $\{a'_0, b'_1, b'_2\}$ with $b'_1 \in B'[b_1, r']$ and $b'_2 \in B'(y_1, b_2)$, which separates $B'(b'_1, b'_2)$ from $\{a_0, b_1, b_2\}$. Since y_1, b_2, b'_2 are incident with a common finite face of G'_0, then a'_0, b_2 are incident with a common finite face of G'_0, and so $\{b'_1, a'_0, b_2\}$ is a 3-cut in G'_0. Moreover, since y_1 is not a cut vertex in G'_0, then $a'_0 \neq a_0$. But now, by (iv) of Lemma 5.0.2, $b'_1 \notin B'(r_1, r']$, and therefore, $b'_1 \in B'[b_1, r_1]$. Now, by Lemma 4.0.7, $b'_1 = b_1$. Then $\{b_1, b_2, a'_0\}$ is a cut in G separating a_0 from $\{a_1, a_2\}$, a contradiction.

Now, by the existence of A_0, B_2, $A'[a_1, x_1] \cup X_0 \cup A'[q, a_2] \cup Q \cup A_0$ and $B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, z_1] \cup Z'_1 \cup B_2$ show that γ is feasible, a contradiction. Thus, y_1 is a cut vertex in G'_0, and $\alpha(A', B') \leq 1$.

Next, we show that $\alpha(A', B') = 1$. We let A^*_0 be the path from a_0 to y_1 in G'_0, internally disjoint from B'. When $y_2 = b_2$, we let $B^* := A'[a_1, x_1] \cup X_1$; when $y_2 \neq b_2$, by Lemma 5.0.8, $x_1 b_2 \in E(G)$, and we let $B^* := A'[a_1, x_1] \cup x_1 b_2$. Then combined with Lemma 3.0.1, the path $A^*_0 \cup B'[q', y_1] \cup Q \cup A'[q, a_2]$ from a_0 to a_2, the path $B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, y_2] \cup B'[y_2, b_2]$ from b_1 to b_2, and the path B^* from a_1 to b_2 show that $\alpha(A', B') = 1$.

Finally, we prove (ii) holds. For otherwise, by planar structure of G'_0, there exists a 2-cut $\{a'_0, b'_1\}$ with $a'_0 \in A'_0$ and $b'_1 \in B'(b_1, r']$, which separates b_1 from q'. Since y_1 is a cut vertex of G'_0, then $\{a'_0, b'_1, b_2\}$ is a 3-cut in G'_0 separating $B'[b'_1, b_2]$ from $\{a_0, b_1, b_2\}$. By Lemma 4.0.7, $b'_1 \notin B'(b_1, r_1]$, and so $b'_1 \in (r_1, r']$. But, by (iv) of Lemma 5.0.2, $a'_0 = a_0$, which implies that G'_0 has no path from a_0 to b_1, internally disjoint from B', and so $\alpha(A', B') = 0$, a contradiction. □

Lemma 5.0.11 *Suppose (i) of Lemma 5.0.8 does not hold and (ii) of Lemma 5.0.8 holds, then* $N(y_1) \cap V(H_0) \subseteq V(W[w_1, w_2])$.

Proof. Since $z_1 \notin V(X_2)$ (by Lemma 5.0.7), then $z_1 \notin W[w_m, y_2]$. So, by (ii) of Lemma 5.0.8, we may assume that there exists $j \in [m-1]$, such that $z_1, z_2 \in W[w_j, w_{j+1}]$ with $z_2 \in W(w_j, w_{j+1}]$. We may also assume $z_2 \notin W[w_1, w_2]$ and $j \neq 1$; otherwise, $N(y_1) \cap V(H_0) \subseteq V(W[w_1, w_2])$.

Since (i) of Lemma 5.0.8 does not hold and $z_2 \notin W[w_1, w_2]$, then we may assume $H_0 - W[x_1, w_1] - W[z_2, w_m]$ contains a path Y_2 from y_2 to w_2.

We may assume $b_2 = y_2$. For, suppose $b_2 \neq y_2$. Then by Lemma 5.0.8, G has an edge from b_2 to x_1, and $a_1 \neq x_1$. Let $e = a_1 b \in E(G)$ with $b \in B'(b_1, r_1]$. Now, combined with Lemma 3.0.1, the path $A'_0 \cup y_1 z_2 \cup W[z_2, w_m] \cup w_m x_2$ from a_0 to a_2, the path $B'_1 \cup Q \cup A'[w_1, q] \cup W[w_1, w_2] \cup Y_2 \cup B'[y_2, b_2]$ from b_1 to b_2, the path $e \cup B'[b_1, b]$ from a_1 to b_1, and the path $A'[a_1, x_1] \cup x_1 b_2$ from a_1 to b_2 show that $\alpha(A', B') = 2$, a contradiction to (i) of Lemma 5.0.10.

Now, we distinguish two cases.

Case 1. $u_2 = x_2$.

(1.1) There does not exist a cross C, D from $c, d \in A'[x_1, x_2)$ to $c', d' \in B'[b_1, y_1]$, such that, $c \in A'[a_1, w_1)$, a_1, c, d, a_2 occur on A' in order, and C, D are internally disjoint from A', B', H'.

For otherwise, such a cross C, D, together with the path $y_1 z_2 \cup W[z_2, w_m] \cup w_m x_2$ from y_1 to x_2 and the path $Y_2 \cup W[w_2, w_1]$ from y_2 to w_1, forces a doublecross. □

(1.2) G has an A'-B' path T from $t \in A'[a_1, w_1)$ to $t' \in B'[b_1, y_1]$, internally disjoint from H'.

For, suppose not, then $\{a_1, w_1, x_2, y_1, y_2\}$ is a 5-cut in G separating $V(H_0)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

We further choose T so that $B'[b_1, t']$ is minimal, and subject to this, $A'[a_1, t]$ is minimal.

(1.3) $t' \in B'[b_1, r']$, $V(T \cap Q) = \emptyset$, and G has no A'-B' path from $A'[a_1, t)$ to $B'[b_1, y_1]$, internally disjoint from H'.

We first prove $t' \in B'[b_1, r']$. For, suppose $t' \in B'(r', y_1]$. Then by the choice of T, we may assume T, R are disjoint, and $r \in A'[w_1, q)$. But then, T, R form a cross, contradicting (1.1).

We may assume $V(T \cap Q) = \emptyset$; otherwise, T, Q are contained in a same A'-B' bridge, then by $w_1 \in A'(t, q)$, the path from w_1 to y_2 in $H' - y_1$, contradicting (v) of Lemma 4.0.9.

Finally, suppose G has an A'-B' path S from $s \in A'[a_1, t)$ to $s' \in B'[b_1, y_1]$, internally disjoint from H'. Then by the choice of T, we have S, T are disjoint, and $s \in B'(t', y_1]$. But then, T, S form a cross, contradicting (1.1). □

(1.4) $H_0 - A'[x_1, t] \cup X_1[x_1, y_2) \cup W[z_1, w_j]$ contains a path Y_2' from y_2 to w_1.

For otherwise, by the planar structure of H_0, there exist $c_1 \in W[z_1, w_j]$ and $c_2 \in A'[x_1, t] \cup X_1[x_1, y_2)$, such that $\{c_1, c_2\}$ is a cut in H_0 separating y_2 from w_1. We notice that $j < m$ and $z_1 \notin V(X_2)$, and so $z_1 \in W[w_j, w_m)$. We may assume $c_2 \notin X_1[x_1, y_2)$; otherwise, $\{c_1, c_2, y_1, y_2, x_2\}$ is a cut in G separating w_m from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. So, $t \in A'(x_1, w_1)$ and $c_2 \in A'(x_1, t]$. But then, by (1.3), G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{x_1, y_2, x_2, y_1, c_1, c_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(X_1 \cup X_2) \subseteq V(G_2)$, and $(G_2, x_1, y_2, x_2, y_1, c_1, c_2)$ is planar, which contradicts Lemma 2.0.3. □

Now, combined with Lemma 3.0.1, the path $A_0' \cup y_1 z_1 \cup W[z_1, w_j] \cup w_j x_2$ from a_0 to a_2, the path $B_1' \cup Q \cup A'[w_1, q] \cup Y_2'$ from b_1 to b_2, the path $A'[a_1, t] \cup T \cup B'[b_1, t']$ from a_1 to b_1, and the path $A'[a_1, x_1] \cup X_1$ from a_1 to b_2 show that $\alpha(A', B') = 2$, a contradiction to (i) of Lemma 5.0.10. □

Case 2. $u_2 \neq x_2$.

(2.1) G has no A'-B' path from a_2 to $B'(b_1, r']$.

For, suppose G has an A'-B' path S from a_2 to $s' \in B'(b_1, r']$. Then, combined with (ii) of Lemma 5.0.10, $A'[a_1, r] \cup R \cup B'[s', r'] \cup S \cup a_2w_m \cup W[w_m, z_2] \cup z_2y_1 \cup A'_0$ and $B'_1 \cup Q \cup A'[q, u_2] \cup u_2w_2 \cup Y_2$ show that γ is feasible, a contradiction. □

(2.2) There does not exist a cross C, D from $c, d \in A'[x_1, x_2)$ to $c', d' \in B'[b_1, y_1]$, such that a_1, c, d, a_2 occur on A' in order, and C, D are internally disjoint from A', B', H'.

For, suppose such a cross exists. We claim that $c \notin A'[a_1, u_2)$; otherwise, such a cross C, D, together with the path $y_1z_2 \cup W[z_2, w_m] \cup w_mx_2$ from y_1 to x_2 and the path $Y_2 \cup w_2u_2$ from y_2 to u_2, forces a doublecross.

So, $d \in A'(u_2, x_2)$. Then, by Lemma 5.0.3, D is an edge with $d' = r'$. Moreover, by (2.1), $b_1 = r'$.

Now, we may assume G has no A'-B' path from $A'[a_1, u_2)$ to $B'(b_1, y_1]$, internally disjoint from H'; otherwise, such a path together with D forms a cross, contradicting our claim that $c \notin A'[a_1, u_2)$. But then, combined with Lemma 5.0.3, $\{b_1, b_2, y_1, u_2, a_2\}$ is a cut in G separating a_1 from a_0, a contradiction. □

(2.3) $H_0 - A'(x_1, w_1] - W[z_2, y_2]$ has a path X' from x_1 to w_j.

For otherwise, by planarity of H_0, there exist $c_1 \in A'(x_1, w_1]$ and $c_2 \in W[z_2, y_2]$, such that c_1, c_2 are incident with a common finite face of H_0, and $\{c_1, c_2\}$ is a cut in H_0 separating x_1 from w_j. But then, (i) of Lemma 5.0.8 holds, a contradiction. □

(2.4) $H_0 - A'[x_1, w_1] \cup X_1[x_1, y_2) \cup W[z_2, w_m)$ contains a path Y_2^* from y_2 to w_2.

For otherwise, by planarity of H_0, there exist $c_1 \in W[z_2, w_m)$ and $c_2 \in A'[x_1, w_1] \cup X_1[x_1, y_2)$, such that c_1, c_2 are incident with a common finite face of H_0. Clearly, $c_2 \notin A'[x_1, w_1]$; otherwise, (i) of Lemma 5.0.8 holds, a contradiction. So $c_2 \in X_1[x_1, y_2)$.

Now, let $w_i \in W(c_1, y_2)$ such that i is minimum. Then we may assume G has an A'-B' path S from $s \in A'(u_i, x_2)$ to $s' \in B'[b_1, y_1]$, internally disjoint from H'; otherwise, $\{u_i, c_1, c_2, y_2, x_2\}$ is a cut in G separating w_m from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

By Lemma 5.0.3, S is an edge with $s' \in \{r', y_1\}$. Now, if $s' = y_1$, then $A'[a_1, w_1] \cup W[w_1, z_1] \cup z_1y_1 \cup A'_0 \cup s's \cup A'[s, a_2]$ and $B'[b_1, q'] \cup Q \cup A'[q, u_i] \cup u_iw_i \cup W[w_i, y_2]$ show that γ is feasible, a contradiction. So $s' = r'$. But then, S, Q form a cross, contradicting (2.2). □

(2.5) z_1, x_2 are incident with a common finite face of $H' - y_1$.

For otherwise, we may assume there exists some $k \in \{j+1, \cdots, m\}$, such that G has an edge from w_k to $u_k \in A'[u_2, x_2)$. We further choose k so that k is minimum, and so $k = j+1$ or $k = j+2$.

Now, we claim that G has no A'-B' path from a_2 to $B'[b_1, y_1]$, internally disjoint from H'. For, suppose G has an A'-B' path S from a_2 to $s' \in B'[b_1, r']$. By (2.1), $s' \notin B'(b_1, r']$. We may also assume $s' \notin B'(r', y_1]$; otherwise, S together with $R, u_kw_k \cup W[w_k, y_2]$, and $X' \cup W[w_j, z_1] \cup z_1y_1$ forces a doublecross. So $s' = b_1$. Moreover, since G has no edge from a_2 to b_1, then S is not an edge, and so $s' = r_1 = b_1$ and S is contained in an A'-B' bridge N with extreme hands n_1, n_2 and feet n'_1, n'_2. Since $s' = b_1$, we have $n'_1 = n'_2 = b_1$. By Lemma 5.0.3, $V(N \cap A'(u_2, x_2)) = \emptyset$. By (v) of Lemma 4.0.9, $n_1 \notin A'[a_1, u_2)$, and so $n_1 = u_2$. But then, $\{n_1, n_2, b_1\}$ is a cut in G separating $V(N)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Now, since the degree of a_2 in G is at least 4, then we may assume G has an edge from a_2 to $w \in W[w_k, w_m)$. But then, combined with Lemma 3.0.1, the path $A'[a_1, r] \cup R \cup B'[b_1, r']$ from a_1 to b_1, the path $A'[a_1, x_1] \cup X_1$ from a_1 to b_2, the path $B'_1 \cup Q \cup A'[q, u_2] \cup u_2w_2 \cup Y_2^*$ from b_1 to b_2, and the path $a_2w \cup W[w, z_2] \cup z_2y_1 \cup A'_0$ from a_2 to a_0 show that $\alpha(A', B') = 2$, a contradiction to (i) of Lemma 5.0.10. □

(2.6) G has two disjoint A'-B' paths from $A'(x_1, v_j)$ to $B'[b_1, y_1]$, internally disjoint from H'.

For otherwise, there exists a vertex $v \in V(G)$, such that $G - v$ does not contain any A'-B' paths from $A'(x_1, v_j)$ to $B'[b_1, y_1]$, internally disjoint from H'. But then, combined with

(2.6), G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{v, x_1, y_2, x_2, u, v_j\}$ with $u = y_1$ (when $z_1 \neq z_2$) or $u = z_1$ (when $z_1 = z_2$), $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, and $V(A'[x_1, v_j] \cup X_1) \subseteq V(G_2)$.

Now, we claim that $(G_2, v, x_1, y_2, x_2, u, v_j)$ is planar, and so Lemma 2.0.3 applies. Obviously, when $v \in A'$, this claim is true. So we may assume $v \notin A'$. Furthermore, if all the A'-B' paths from v to A' are edges, then our claim is still true. Therefore, we may assume there exists some A'-B' bridge N with feet n_1', n_2' and extreme hands n_1, n_2, such that $v \in N$ and N contains a path P^* from v to $A'[n_1, n_2]$, which is not an edge and internally disjoint from A'. By (v) of Lemma 4.0.9, $H' - y_1$ does not contain a path from $A'(n_1, n_2)$ to y_2, internally disjoint from A'. Hence, $\{n_1, n_2, v\}$ is a cut in G separating $V(P^*)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. $\square$

Now, we let T_1, T_2 be disjoint A'-B' paths from $t_1, t_2 \in A'(x_1, u_j)$ to $t_1', t_2' \in B'[b_1, y_1]$, respectively, internally disjoint from H', such that a_1, t_1, t_2, a_2 occur on A' in order, subject to this, $B'[t_1', t_2']$ is maximal, and subject to this, $A'[t_1, t_2]$ is maximal. By (2.2), b_1, t_1', t_2', b_2 occur on B' in order.

(2.7) $t_1' \in B'[b_1, r']$.

For otherwise, $t_1' \in B'(r', y_1]$. We may first assume R is internally disjoint from T_1, T_2. For otherwise, let $v \in V(R \cap (T_1 \cup T_2))$, such that $R[r', v]$ is minimal. If $v \in V(T_1)$, then $R[r', v] \cup T_1[v, t_1], T_2$ contradict the choice of T_1, T_2; if $v \in V(T_2)$, then $T_1, R[r', v] \cup T_2[v, t_2]$ form a cross, contradicting (2.2).

Now, if $r \in A'[a_1, t_1]$, then R, T_2 contradict the choice of T_1, T_2; if $r \in A'(t_1, q)$, then R, T_1 form a cross, contradicting (2.2). $\square$

(2.8) There exist $c_1, c_2 \in V(G_0')$, such that $c_1 \in B'[b_1, t_1']$, $c_2 \in B'[t_2', y_1]$, and c_1, c_2 are incident with a common finite face of G_0'.

In fact, due to the existence of the path $A'[a_1, x_1] \cup X' \cup w_j v_j \cup A'[v_j, a_2]$ from a_1 to a_2 and the path $B'[b_1, t_1'] \cup T_1 \cup A'[t_1, t_2] \cup T_2 \cup B'[t_2', y_1] \cup y_1 z_2 \cup W[z_2, y_2]$ from b_1 to b_2,

G'_0 contains no path from a_0 to $B'(t'_1, t'_2)$, internally disjoint from B' (by Lemma 3.0.2). Hence, (2.8) holds. □

Now, we further choose c_1, c_2 so that $B'[c_1, c_2]$ is maximal.

(2.9) $G'_0 - B'(b_1, c_1) \cup B'[c_2, y_1] \cup A'_0$ contains a path B'_0 from b_1 to c_1.

For otherwise, $V(B'(b_1, c_1)) \neq \emptyset$, and by planarity of G'_0, we may assume there exist $b'_1 \in B'(b_1, c_1)$ and $a'_0 \in B'[c_2, y_1] \cup A'_0$, such that b'_1, a'_0 are incident with a common finite face of G'_0. Now, if $a'_0 \in B'[c_2, y_1]$, then b'_1, a'_0 contradict the choice of c_1, c_2; if $a'_0 \in A'_0$, then $\{b'_1, a'_0, b_2\}$ is a 3-cut in G'_0, contradicting Lemma 4.0.7. □

(2.10) $G'_0 - B'(b_1, c_2) \cup B'(c_2, y_1] \cup A'_0$ contains a path B''_0 from b_1 to c_2.

For otherwise, by planarity of G'_0, we may assume there exist $b'_1 \in B'(b_1, c_2)$ and $a'_0 \in B'(c_2, y_1] \cup A'_0$, such that b'_1, a'_0 are incident with a common finite face of G'_0. Now, if $a'_0 \in B'(c_2, y_1]$, then b'_1, a'_0 or c_1, a'_0 contradict the choice of c_1, c_2. So $a'_0 \in A'_0$. We may further assume $b_1 = c_1$ and $b'_1 \in B'(c_1, c_2)$; otherwise, $\{b'_1, a'_0, b_2\}$ or $\{c_1, a'_0, b_2\}$ is a 3-cut in G'_0, contradicting Lemma 4.0.7. But now, since a_0, b_1, b'_1, c_2 are incident with a common finite face of G'_0, then $\alpha(A', B') = 0$, a contradiction to (i) of Lemma 5.0.10. □

(2.11) G has no A'-B' path from $B'(b_1, c_1)$ to A'.

For otherwise, since $c_1 \in B'[b_1, t'_1]$ and $t'_1 \in B'[b_1, r_1]$, then $c_1 \in B'[b_1, r_1]$, and so such an A'-B' path from $B'(b_1, c_1)$ to A' should be an edge e from $b \in B'(b_1, c_1)$ to $a \in A'[a_1, x_1] \cup \{a_2\}$. By (2.2), $a \neq a_2$, and so $a \in A'[a_1, x_1]$.

But then, combined with Lemma 3.0.1, the path $A'[a_1, a] \cup e \cup B'[b_1, b]$ from a_1 to b_1, the path $A'[a_1, x_1] \cup X_1$ from a_1 to b_2, the path $A'_0 \cup B'[q', y_1] \cup Q \cup A'[q, a_2]$ from a_0 to a_2, and the path $B'_0 \cup B'[c_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, y_2]$ from b_1 to b_2 show that $\alpha(A', B') = 2$, a contradiction to (i) of Lemma 5.0.10. □

(2.12) G has an A'-B' path T from $t' \in B'(c_2, y_1)$ to $t \in A'[x_1, x_2]$.

For otherwise, combined with (2.11), $\{b_1, c_1, c_2, y_1, b_2\}$ is a cut in G separating a_0 from $\{a_1, a_2\}$, a contradiction. □

Now, we choose T so that $A'[t, a_2]$ is minimal.

(2.13) $t \neq a_2$.

For otherwise, combined with Lemma 3.0.1, the path $A'[a_1, t_1] \cup T_1 \cup B'[b_1, t_1']$ from a_1 to b_1, the path $A'[a_1, x_1] \cup X_1$ from a_1 to b_2, the path $T \cup B'[t', y_1] \cup A_0'$ from a_2 to a_0, and the path $B_0'' \cup B'[t_2, c_2] \cup T_2 \cup A'[t_2, u_2] \cup u_2w_2 \cup W[w_2, y_2]$ from b_1 to b_2 show that $\alpha(A', B') = 2$, a contradiction to (i) of Lemma 5.0.10. □

(2.14) T is internally disjoint from T_1, T_2, and $t = u_2 = v_j$.

First, we may assume T is internally disjoint from T_1, T_2. For otherwise, let $v \in V(T \cap (T_1 \cup T_2))$, such that $T[v, t']$ is minimal. Now, if $v \in T_1$, then $T_1[t_1, v] \cup T[v, t'], T_2$ form a cross, contradicting (2.2); if $v \in T_2$, then $T_1, T_2[t_2, v] \cup T[v, t']$ contradict the choice of T_1, T_2.

Now, by (2.2), we may assume $t \in A'[t_2, a_2)$. By the choice of T_1, T_2, we may further assume $t \notin A'[t_2, v_j)$. Finally, by Lemma 5.0.3, we have $t \notin A'(u_2, a_2)$, and so $t = u_2 = v_j$. □

(2.15) $t_1 \in A'[a_1, w_1)$.

For otherwise, $t_1 \in A'[w_1, v_j)$. We first claim G has an A'-B' path T_0 from $t_0 \in A'(x_1, w_1)$ to $t_0' \in B'[b_1, y_1]$, internally disjoint from H'. For otherwise, by (2.5) and $u_2 = v_j$, G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{x_1, w_1, u_2, u, x_2, y_2\}$ with $u = y_1$ (when $z_1 \neq z_2$) or $u = z_1$ (when $z_1 = z_2$), $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(X_1 \cup X_2) \subseteq V(G_2)$, and $(G_2, x_1, w_1, u_2, u, x_2, y_2)$ is planar, a contradiction to Lemma 2.0.3.

We may assume T_0 is disjoint from T_1, T_2. For otherwise, let $v \in V(T_0 \cap (T_1 \cup T_2))$, such that $T_0[v, t_0']$ is minimal. Now, if $v \in T_1$, then $T_1[t_1, v] \cup T_0[v, t_0'], T_2$ contradict the choice of T_1, T_2; if $v \in T_2$, then $T_1, T_2[t_2, v] \cup T_0[v, t_0']$ form a cross, contradicting (2.2).

But then, either T_0, T_2 contradict the choice of T_1, T_2, or T_0, T_1 form a cross, contradicting (2.2). $\square$

Now, combined with (2.15) and Lemma 3.0.1, the path from $A'[a_1, t_1] \cup T_1 \cup B'[b_1, t_1']$ from a_1 to b_1, the path $A'[a_1, x_1] \cup X_1$ from a_1 to b_2, the path $A'[t, a_2] \cup T \cup B'[t', y_1] \cup A_0'$ from a_2 to a_0, and the path $B_0'' \cup B'[t_2, c_2] \cup T_2 \cup A'[w_1, t_2] \cup W[w_1, y_2]$ from b_1 to b_2 show that $\alpha(A', B') = 2$, a contradiction to (i) of Lemma 5.0.10. $\square$

Lemma 5.0.12 *None of (i) and (ii) of Lemma 5.0.8 holds.*

Proof. For, suppose (i) or (ii) of Lemma 5.0.8 holds.

(i) When (i) of Lemma 5.0.8 holds, by Lemma 5.0.9, we may choose 2-cut $\{z_1', z_2'\}$ so that $z_1' \in A'[x_1, w_1]$.

(ii) When (ii) of Lemma 5.0.8 holds, by Lemma 5.0.11, we may assume $N(y_1) \cap V(H_0) \subseteq V(W[w_1, w_2])$. For notation convenience, we let $z_1' := w_1$ and $z_2' := z_1$.

(1) $z_2' \notin V(X_2)$.

For, suppose $z_2' \in V(X_2)$. Since $z_1 \notin V(X_2)$ by Lemma 5.0.7, we may assume (i) holds. Then $z_1' = x_1$; or else, it contradicts Lemma 5.0.1 that $H' - y_1 - A'(x_1, x_2)$ contains disjoint paths from $N(y_1) - V(A')$, y_2 to x_1, x_2, respectively. But now, $\{x_1, y_2, z_2'\}$ is a cut in G separating $V(X_1(x_1, y_2))$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. $\square$

Since $z_2' \notin V(X_2)$, then $w_m \in W(z_2', y_2)$. Now, we let $h \in \{2, \cdots, m\}$, such that $w_h \in W(z_2', y_2)$, and subject to this, h is minimum.

(2) $H_0 - W[w_h, y_2] \cup A'(x_1, w_1]$ contains a path Y from $N(y_1) - V(A')$ to x_1.

Let $v \in N(y_1) \cap V(H_0)$, such that $v \notin V(W[w_h, y_2] \cup A'(x_1, w_1])$. For, suppose such a path Y does not exist. Then, combined with the planar structure of H_0, there exist $z_1'' \in A'[x_1, z_1']$, $z_2'' \in W[w_h, y_2]$, such that z_1'', z_2'' are incident with a common finite face of H, and $\{z_1'', z_2''\}$ is a 2-cut in H_0. Hence, (i) holds. But then, $\{z_1'', z_2''\}$ contradicts the choice of $\{z_1', z_2'\}$. $\Box$

Now, we let Y_1 be the path obtained from Y by adding the vertex y_1 and the edge from y_1 to the end of Y^*, such that Y_1 is a path from y_1 to x_1, and let $Y_2 := W[y_2, w_h] \cup w_h u_h$, and so Y_2 is a path from y_2 to u_h, disjoint from Y_1.

(3) $H_0 - W[z_2, w_m]$ has a path Y_2' from y_2 to z_1', internally disjoint from A'.

For otherwise, by the planar structure of H_0, there exist $z_1'' \in A'[x_1, z_1'), z_2'' \in W[z_2, w_m]$, such that z_1'', z_2'' are incident with a common finite face of H_0, and $\{z_1'', z_2''\}$ is a 2-cut in H_0. Hence, (i) holds. But then, $\{z_1'', z_2''\}$ contradicts the choice of $\{z_1', z_2'\}$. $\Box$

Now, we let $Y_1' := Z_2' \cup W[z_2, w_m] \cup w_m v_m$, and so Y_1' is a path from y_1 to x_2, disjoint from Y_2'.

(4) G has no cross C, D from $c, d \in A'$ to $c', d' \in B'[b_1, y_1]$, such that C, D are internally disjoint from A', B', H', $c \in A'[a_1, z_1')$, and $d \in A'(c, x_2)$.

For otherwise, C, D, Y_1', Y_2' force a doublecross, a contradiction. $\Box$

(5) If $u_h \neq x_2$ and G has an A'-B' path S from $s \in A'(u_h, x_2]$ to $s' \in B'[b_1, y_1]$, internally disjoint from H', then $b_1 = r_1 = r' = s'$ and S is an edge from s to s'.

We may first assume S and R are disjoint; otherwise, S, R are contained in an A'-B' bridge, which contradicts (v) of Lemma 4.0.9 due to the path $u_h w_h \cup W[w_h, y_2]$ from u_h to y_2.

Now, $s' \notin B'(r', y_1]$; otherwise, S, R, Y_1, Y_2 form a doublecross by $u_h \neq x_2$. Thus, G has no A'-B' path from $A'(u_h, x_2]$ to $B'(r', y_1]$, which further implies that S, Q are disjoint.

We may assume $b_1 = r_1$ and so $a_1 = x_1$ by (iii) of Lemma 5.0.4. For, suppose $b_1 \neq r_1$. Then $s' \neq b_1$; otherwise, $s = x_2 = a_2$, and S is an edge from a_2 to b_1, a contradiction. So $s' \in B'(b_1, r']$. But then, $A'[a_1, r] \cup R \cup B'[s', r'] \cup S \cup A'[s, a_2] \cup Y_1 \cup A_0'$ and

$B'_1 \cup Q \cup A'[q, u_h] \cup Y_2 \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction. (B'_1, A'_0 are defined in (ii) of Lemma 5.0.10.)

We may also assume $r_1 = r'$. For, suppose $r_1 \neq r'$. By Lemma 5.0.2, there exist a core H' of A, B with r_1, r_2 as feet and $r' \in B'(r_1, r_2)$, and an A'-B' bridge M_4 with extreme hands l_4, r_4 and feet l'_4, r'_4, such that R is internally disjoint from M_4, $l_4 = r_4 = x_i$ for some $i \in [2]$, and $r' \in B'(l'_4, r'_4)$. Since G has no A'-B' path from $A'(u_h, x_2]$ to $B'(r', y_1]$, then $i \neq 2$ and S is internally disjoint from M_4. Next, $s' \neq r'$. For, suppose $s' = r'$. Let P^* be the path from l'_4 to r'_4 in M_4, internally disjoint from A', B', then $A'[a_1, r] \cup R \cup S \cup A'(u_h, a_2] \cup Y_1 \cup A'_0$ and $B'[b_1, l'_4] \cup P^* \cup B'[r'_4, q'] \cup Q \cup A'[q, u_h] \cup Y_2$ show that γ is feasible, a contradiction. So $s' \in B'[r_1, r')$ and $s = x_2$ (by the definition of r'). Now, we see that $s' \notin B'(r_1, r')$ and S is not contained in an A'-B' bridge. For otherwise, by Lemma 4.0.9, S is contained in H', which further implies x_2 is an extreme hand of H'. So H' is a main core of A, B, a contradiction to Lemma 4.0.8. Therefore, $s' = r_1$, and S is an edge from b_1 to x_2, which implies $a_2 \neq x_2$, a contradiction.

Thus, now, $b_1 = r_1 = r' = s'$. To finish (5), we just need to prove that S is an edge from $A'(u_h, x_2]$ to s'. For otherwise, S is contained in an A'-B' bridge N with extreme hands n_1, n_2. Obviously, $V(N \cap B') \subseteq \{b_1, y_1\}$ (by $s' = r'$). Moreover, by Lemma 5.0.3, $V(N \cap A'(u_h, x_2)) = \emptyset$. Hence, $n_1 \in A'[x_1, u_h]$ and $n_2 = x_2$. By (v) of Lemma 4.0.9, $H' - y_1$ does not have a path from $A'(n_1, n_2)$ to y_2, internally disjoint from A'. So, by the existence of path Y_2, $n_1 \notin A'[x_1, u_2)$. So $n_1 = u_h$. But then, $\{n_1, n_2, b_1, y_1\}$ is a cut in G separating $V(N)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

(6) $x_1 \neq z'_1$.

For, suppose $x_1 = z'_1$. Since $w_1 \neq x_1$, then (i) holds. And G has an A'-B' path from $A'(u_h, x_2)$ to $B'[b_1, y_1]$ internally disjoint from $H' - y_1$; otherwise, G has a separation (G_1, G_2) of order 5, such that $V(G_1 \cap G_2) = \{x_1, z'_2, u_h, x_2, y_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, and $V(X_1 \cup X_2) \subseteq V(G_2)$, a contradiction. Hence, $A'(u_h, x_2) \neq \emptyset$, and by (5), $b_1 = r_1 = r'$, and $a_1 = x_1$ (by (iii) of Lemma 5.0.4). But then, G has a separation

(G_1, G_2) of order 6, such that $V(G_1 \cap G_2) = \{x_1, z_2', u_h, x_2, y_2, b_1\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(X_1 \cup X_2) \subseteq V(G_2)$, and $(G_2, x_1, y_2, x_2, b_1, u_h, z_2')$ is planar, a contradiction to Lemma 2.0.3. □

(7) $b_2 = y_2$.

For, suppose $b_2 \neq y_2$. By Lemma 5.0.8, $N(b_2) = \{y_2, x_1\}$ and $a_1 \neq x_1$. Now, let $e' = a_1 b' \in E(G)$ with $b' \in B'(b_1, r_1] \cup B'[y_2, b_2)$. By (i) of Lemma 5.0.4, $b' \in B'(b_1, r_1]$. Now, since $x_1 \neq z_1'$, combined with Lemma 3.0.1, the path $A_0' \cup Y_1' \cup A'[x_2, a_2]$ from a_0 to a_2, the path $B_1' \cup Q \cup A'[z_1', q] \cup Y_2' \cup B'[y_2, b_2]$ from b_1 to b_2, the path $e' \cup B'[b_1, b']$ from a_1 to b_1, and the path $A'[a_1, x_1] \cup e$ from a_1 to b_2 show that $\alpha(A', B') = 2$, a contradiction to (i) of Lemma 5.0.10. □

(8) G has an A'-B' path from $A'[a_1, z_1')$ to $B'(b_1, y_1]$, internally disjoint from H'.

For, suppose G has no A'-B' path from $A'[a_1, z_1')$ to $B'(b_1, y_1]$, internally disjoint from H'. Then $a_1 = x_1$. Now, by (5) and (7), when (i) holds, $\{b_1, b_2, z_1', z_2', u_h\}$ is a cut in G separating a_1, a_2 from a_0, a contradiction; when (ii) holds, $\{b_1, b_2, z_1', y_2, u_h\}$ (when $z_1 \neq w_2$) or $\{b_1, b_2, z_1', z_1, u_h\}$ (when $z_1 = w_2$) is a cut in G separating a_1, a_2 from a_0, a contradiction. □

(9) If $u_h \neq x_2$, then G has no A'-B' path from $A'(u_h, x_2]$ to $B'[b_1, y_1]$, internally disjoint from H'.

Suppose G has an A'-B' path S from $s \in A'(u_h, x_2]$ to $s' \in B'[b_1, y_1]$, internally disjoint from H'. Then by (5), S is an edge from s to b_1 with $b_1 = r_1 = r' = s'$. So $s \neq a_2$, and $s \in A'(u_h, a_2)$.

By (8), G has an A'-B' path from $A'[a_1, z_1')$ to $B'(b_1, y_1]$, internally disjoint from H'. Now, this path together with S forms a cross, which contradicts (4). □

(10) G has disjoint A'-B' paths from $A'[a_1, z_1')$ to $B'[b_1, y_1]$, internally disjoint from H'.

For otherwise, there exists a vertex $v \in V(G)$, such that $G - v$ has no A'-B' path from $A'[a_1, z_1')$ to $B'[b_1, y_1]$, internally disjoint from H'. Then combined with (10), there exists a separation (G_1, G_2) in G of order 4, such that $V(G_1 \cap G_2) = \{v, z_1', z_2', u_h\}, a_0, y_1 \in V(G_1 - G_2)$, and $a_1, a_2, b_2 \in V(G_2)$.

Now, we claim that $(G_2, v, z_1', z_2', u_h, a_2, b_2, a_1)$ is planar. Obviously, when $v \in A'$, this claim is true. So we may assume $v \notin A'$. Furthermore, if $v \in B'$ and all the A'-B' paths from v to A' are edges, then our claim is still true. Therefore, we may assume there exists some A'-B' bridge N with feet n_1', n_2' and extreme hands n_1, n_2, such that $v \in N$. By (v) of Lemma 4.0.9, $H' - y_1$ does not contain a path from $A'(n_1, n_2)$ to y_2, internally disjoint from A'. Now, $v \notin B'$; otherwise, $n_1' = n_2' = v$, and $\{n_1, n_2, v\}$ is a cut in G separating $V(N)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. So $v \notin V(A' \cup B')$. Now, N has a separation (N', N'') of order 1, such that $V(N' \cap N'') = \{v\}, n_1, n_2 \in V(N' - N'')$, and $n_1', n_2' \in V(N'' - N')$. We see that $V(N') = \{n_1, n_2, v\}$; or else, $\{n_1, n_2, v\}$ is a cut in G separating $V(N') - \{n_1, n_2, v\}$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. So, $V(N') = \{n_1, n_2, v\}$, N' is planar, and $(G_2, v, z_1', z_2', u_h, a_2, b_2, a_1)$ is planar. So our claim is true.

Now, we see that if $v = a_1, u_h = a_2$, then $\{v, z_1', z_2', u_h, b_2\}$ is a cut in G separating $V(X_1 \cup X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction; if $v \neq a_1, u_h = a_2$ or $v = a_1, u_h \neq a_2$, then Lemma 2.0.3 applies; if $v \neq a_1, u_h \neq a_2$, then Lemma 2.0.4 applies. $\square$

By (10), we let T_1, T_2 be two disjoint A'-B' paths from $t_1, t_2 \in A'[a_1, z_1')$ to $t_1', t_2' \in B'[b_1, y_1]$, such that T_1, T_2 are internally disjoint from H', a_1, t_1, t_2, a_2 occur on A' in order, and subject to this, $A'[t_1, t_2] \cup B'[t_1', t_2']$ are maximal. By (4), T_1, T_2 do not form a cross, and so b_1, t_1', t_2', y_1 occur on B' in order.

(11) Q is internally disjoint from $T_1, T_2, t_1' \in B'[b_1, r']$, and $t_2' \notin B'(q', y_1]$.

For, suppose Q is not internally disjoint from T_j for some $j \in [2]$, then Q, T_j are contained in a same A'-B' bridge. But then, the existence of the path from z_1' to y_2 in $H' - y_1$ contradicts (v) of Lemma 4.0.9.

Hence, $t'_2 \notin B'(q', y_1]$ (to void the cross contradicting (4)).

We also see that $t'_1 \in B'[b_1, r']$. For otherwise, $t'_1 \in B'(r', t'_2)$. Now, $V(R \cap (T_1 \cup T_2)) = \emptyset$. For, suppose there exists $u \in V(R \cap (T_1 \cup T_2))$, then we choose u so that $R[r', u]$ is minimal. Now, if $u \in T_1$, then $R[r', u] \cup T_1[u, t_1]$ and T_2 contradict the choice of T_1, T_2; if $u \in T_2$, then $T_1, R[r', u] \cup T_2[u, t_2]$ form a cross, contradicting (4). So $V(R \cap (T_1 \cup T_2)) = \emptyset$. But then if $r \in A'[a_1, t_1]$, R and T_2 contradict the choice of T_1, T_2; if $r \in A'(t_1, q)$, then T_1, R form a cross, contradicting (4). $\square$

We let Q_0 be an A'-B' path from $q_0 \in A'(z'_1, a_2]$ to $q'_0 \in B'[b_1, y_1]$, internally disjoint from H', such that $B'[q'_0, y_1]$ is minimal. By the existence of Q, obviously, $q'_0 \in B'[q', y_1]$.

(12) There do not exist $c_1, c_2 \in V(G'_0)$, such that $c_1 \in B'[b_1, t'_1]$, $c_2 \in B'[q'_0, y_1]$, and c_1, c_2 are incident with a common finite face of G'_0.

For, suppose (12) fails. We choose c_1, c_2 so that $B'[c_1, c_2]$ is maximal. Since $t'_1 \in B'[b_1, r']$, then $c_1 \in B'[b_1, r']$. We may further assume $c_1 \in B'[b_1, r_1]$. In fact, by (iii) of Lemma 5.0.2, when $r' \neq r_1$, we have $r' \in B'(r_1, r_2)$, and so r', r_1, r_2 are incident with a common finite face of G'_0, which further implies $c_1 \in B'[b_1, r_1]$ by the choice of c_1, c_2.

Now, we may assume G has an A'-B' path T_3 from $t'_3 \in B'(b_1, c_1) \cup B'(c_2, y_1)$ to $t_3 \in A'$. For otherwise, $\{b_1, b_2, c_1, c_2, y_1\}$ is a cut in G separating a_0 from $\{a_1, a_2\}$, a contradiction.

We may assume $t'_3 \in B'(c_2, y_1)$. For otherwise, $t'_3 \in B'(b_1, c_1)$, and so $t'_3 \in B'(b_1, r_1)$, which further implies T_3 is an edge. Now, by the choice of T_1, T_2, and by (4) and (9), we have $t_3 = u_h = a_2$. Thus, $A'[a_1, t_1] \cup T_1 \cup B'[t'_3, t'_1] \cup T_3 \cup Y'_1 \cup A'_0$ and $B'_1 \cup Q \cup A'[z'_1, q] \cup Y'_2$ show that γ is feasible, a contradiction.

Now, by the choice of Q_0, $t_3 \notin A'(z'_1, a_2]$ and T_3, Q_0 are disjoint. Moreover, by (4), to forbid the cross T_3, Q_0, $t_3 \notin A'[a_1, z'_1)$, and so $t_3 = z'_1$.

We claim that $G'_0 - B'[t'_1, q'_0] - A'_0$ contains a path B^*_3 from b_1 to t'_3. For otherwise, by maximality of $B'[c_1, c_2]$, there exists a vertex $c_3 \in V(A'_0)$, such that $\{c_2, c_3\}$ is a cut in

G_0' separating b_1 from t_3'. Moreover, by the maximality of $B'[c_1, c_2]$, there does not exist an A'-B' bridge with feet n_1', n_2', such that $n_1' \in B'[b_1, c_2)$ and $n_2' \in B'(c_2, y_1]$. Hence, $\{z_1', c_2, c_3, y_1\}$ is a cut in G separating t_3' from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Now, $A'[a_1, t_1] \cup T_1 \cup B'[t_1', q_0'] \cup Q_0 \cup A'[q_0, a_2] \cup Y_1' \cup A_0'$ and $B_3^* \cup T_3 \cup Y_2'$ show that γ is feasible, a contradiction. □

(13) $G_0' - B'(b_1, t_1'] - B'[q_0', y_1] \cup A_0'$ contains a path B_1^* from b_1 to $B'(t_1', q_0')$.

For otherwise, $b_1 \neq t_1'$, and there exist $c_1, c_2 \in V(G_0')$ with $c_1 \in B'(b_1, t_1']$ and $c_2 \in B'[q_0', y_1] \cup A_0'$, such that c_1, c_2 are incident with a common finite face of G_0'. By (12), $c_2 \notin B'[q_0', y_1]$. So we may assume $c_2 \in A_0'$. Since $t_1' \in B'[b_1, r']$, then $c_1 \in B'(b_1, r']$. By Lemma 4.0.7, $c_1 \notin B'(b_1, r_1]$. So $c_1 \in B'(r_1, r']$. But then, by (iv) of Lemma 5.0.2, $c_2 = a_0$ and $b_1 = r_1$. Then $\alpha(A', B') = 0$, a contradiction to (i) of Lemma 5.0.10. □

(14) When (i) holds, $H' - y_1 - V(X_1[x_1, y_2)) \cup \{z_2'\}$ contains a path Y_2^* from z_1' to y_2, internally disjoint from A'.

For otherwise, there exists a vertex $u \in V(A'[x_1, z_1') \cup X_1[x_1, y_2))$, such that u, z_2' are incident with a common finite face of $H' - y_1$. By the choice of $\{z_1', z_2'\}$, $u \notin V(A'[x_1, z_1'))$. So $u \in V(X_1(x_1, y_2))$. But then, $\{u, z_2', u_h, x_2, y_2\}$ is a cut in G separating $V(X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

(15) When (i) holds, $H' - y_1 - A'(x_1, z_1') - W[z_2', y_2]$ contains a path X^* from x_1 to z_1'; when (ii) holds, $H' - y_1 - A'(x_1, z_1') - W[z_2, y_2]$ contains a path X^* from x_1 to z_1'.

For otherwise, let $v = z_2'$ when (i) holds; and let $v = z_2$ when (ii) holds. Then there exists a 2-cut $\{z_1'', z_2''\}$ in H_0, such that $z_1'' \in A'(x_1, z_1')$, $z_2'' \in W[v, y_2]$, and z_1'', z_2'' are incident with a common finite face of H_0. Hence, (i) holds. But then $\{z_1'', z_2''\}$ contradicts the choice of $\{z_1', z_2'\}$. □

(16) G has no A'-B' path from $A'(t_1, z_1']$ to $B'(t_1', q_0')$, disjoint from T_1, Q_0.

For, suppose G has an A'-B' path T from $t \in A'(t_1, z_1']$ to $t' \in B'(t_1', q_0')$, disjoint from T_1, Q_0. When (i) holds, we let B^* be the path from b_1 to b_2 in $B_1^* \cup B'(t_1', q_0') \cup T \cup A'[t, z_1'] \cup Y_2^*$; when (ii) holds, we let B^* be the path from b_1 to b_2 in $B_1^* \cup B'(t_1', q_0') \cup T \cup W'[t, y_2]$. Now, combined with Lemma 3.0.1, the path B^* from b_1 to b_2, the path $A'[q_0, a_2] \cup Q_0 \cup B'[q_0', y_1] \cup A_0'$ from a_2 to a_0, the path $A'[a_1, t_1] \cup T_1 \cup B'[b_1, t_1']$ from a_1 to b_1, and the path $A'[a_1, x_1] \cup X_1$ from a_1 to b_2 show that $\alpha(A', B') = 2$, a contradiction to (i) of Lemma 5.0.10. □

(17) $t_2' = q_0'$.

For otherwise, by (16), $V(T_2 \cap Q_0) \neq \emptyset$. So T_2, Q_0 are contained in a same A'-B' bridge. But the existence of the path from z_1' to y_2 in $H' - y_1$ contradicts (v) of Lemma 4.0.9. □

(18) G has an A'-B' path R^* from r' to $A'(x_1, z_1')$, and $t_1' = r'$.

We may assume G has an A'-B' path from r' to $A'(x_1, z_1')$. For otherwise, R is disjoint from T_2, and R, T_2 form a cross, contradicting (4).

Now, we prove $t_1' = r'$. For otherwise, $r' \in B'(t_1', q_0')$. Now, by (16), $V(R^* \cap (T_1 \cup Q_0)) \neq \emptyset$. Obviously, by the definition of r', $V(R^* \cap T_1) = \emptyset$. Thus, R^*, Q_0 are contained in a same A'-B' bridge. But then, the path from z_1' to y_2 in $H' - y_1$ contradicts (v) of Lemma 4.0.9. □

Now, the path $A'[a_1, x_1] \cup X^* \cup A'[z_1', a_2]$ from a_1 to a_2 and the path $B'[b_1, r'] \cup R \cup A'[r, t_2] \cup T_2 \cup B'[t_2', y_1] \cup Z_2' \cup W[z_2, y_2]$ from b_1 to b_2 show that G_0' does not contain a path from $B'(t_1', t_2')$ to a_0, internally disjoint from B'; or else, it contradicts (i) of Lemma 3.0.2. So, there exist $c_1 \in B'[b_1, t_1']$ and $c_2 \in B'[t_2', y_2]$, such that c_1, c_2 are incident with a common finite face of G_0', a contradiction to (12). □

CHAPTER 6

SLIM CONNECTOR

In this chapter, we deal with the case, when no ideal frame in $\gamma = (G, a_0, a_1, a_2, b_1, b_2)$ admits a fat connector.

Definition. Let A, B be an ideal frame in γ w.r.t. a_0. Assume there does not exist any fat connectors of ideal frame A, B, then let $G_0 := G - A$. By Lemma 2.0.6 and the structure of slim connectors, G_0 has a disk representation with B and a_0 occurring on the boundary of the disk, and any A-B path in γ is induced by a single edge.

Lemma 6.0.1 *Suppose γ is infeasible, and A', B' is a core a_0-frame in γ. Let $a_{-1} := a_2$ and $a_3 := a_0$. Then*

(i) There do not exist $i \in \{0, 1, 2\}$, a graph H and vertices $s, s' \in V(H)$, such that G is obtained from H by identifying s with s', and $(H, a_{i-1}, b_1, a_{i+1}, b_2)$ is planar.

(ii) For any $i \in \{0, 1, 2\}$, $(G - a_{i-1}, a_i, b_1, a_{i+1}, b_2)$ or $(G - a_{i+1}, a_i, b_1, a_{i-1}, b_2)$ is not planar.

(iii) There do not exist a permutation π of $\{0, 1, 2\}$, a graph H and $s, t, s', t' \in V(H)$, such that G is obtained from H by identifying s with s' and t with t', respectively, $(H, a_{\pi(0)}, b_1, a_{\pi(1)}, s, t, s', t', a_{\pi(2)}, b_2)$ is planar, and $a_{\pi(1)}, t, s', a_{\pi(2)}$ are distinct in H.

Proof. Let n denote the number of vertices in G. Obviously, $|E(G)| \geq 3n - 7$.

For, suppose (i) fails, and there exist $i \in \{0, 1, 2\}$, a graph H and vertices $s, s' \in V(H)$, such that G is obtained from H by identifying s with s', and $(H, a_{i-1}, b_1, a_{i+1}, b_2)$ is planar. Obviously, $|E(H)| \geq |E(G)| \geq 3n - 7$. Moreover, we let $H' := H + \{a_{i-1}b_1, a_{i-1}b_2, a_{i+1}b_1, a_{i+1}b_2, b_1b_2\}$, then H' is planar with $|V(H')| = n+1$ and $|E(H')| \geq$

$3n-2=3(n+1)-5$. But now, it contradicts that a planar graph with $n+1$ (≥ 3) vertices has at most $3(n+1)-6$ edges.

For, suppose (ii) fails, and for some $i \in \{0,1,2\}$, $(G-a_{i-1}, a_i, b_1, a_{i+1}, b_2)$ and $(G-a_{i+1}, a_i, b_1, a_{i-1}, b_2)$ are planar. Without loss of generality, we assume $i=0$, and the degree of a_1 in G is no more than the degree of a_2 in G. Let k denote the degree of a_1 in G, and let $G' := G + \{a_2b_1, a_2b_2, a_0b_1, a_0b_2, b_1b_2\}$. Obviously, $G'-a_1$ is planar. We also see that a_2 has degree at least $k+2$ in G', a_0 has degree at least 6 in G', and b_j has degree at least 5 in G' for $j \in [2]$. Moreover, all other vertices of G' not in $\{a_0, a_1, a_2, b_1, b_2\}$ have degree at least 6 in G'. Hence, the sum of degrees of each vertices in $G'-a_1$ is at least $6(n-5)+(k+2)+6+5+5-k=6n-12$. So the number of edges in $G'-a_1$ is at least $3n-6=3(n-1)-3$. But now, it contradicts that a planar graph with $n-1$ (≥ 3) vertices has at most $3(n-1)-6$ edges.

For, suppose (iii) fails due to some permutation π of $\{0,1,2\}$, a graph H and $s,t,s',t' \in V(H)$. Obviously, $|E(H)| \geq |E(G)| \geq 3n-7$. Moreover, we let $H' := H+\{b_1a_{\pi(0)}, b_1a_{\pi(1)}, b_2a_{\pi(0)}, b_2a_{\pi(2)}, a_{\pi(0)}a_{\pi(1)}, a_{\pi(0)}a_{\pi(2)}, a_{\pi(0)}t, a_{\pi(0)}s'\}$. Since G^* is 6-connected and $(H, a_{\pi(0)}, b_1, a_{\pi(1)}, s, t, s', t', a_{\pi(2)}, b_2)$ is planar, then $a_{\pi(0)}a_{\pi(1)}, a_{\pi(0)}a_{\pi(2)}, a_{\pi(0)}t, a_{\pi(0)}s' \notin E(H)$, and so H' is planar with $|V(H')|=n+2$ and $|E(H')| \geq 3n+1 = 3(n+2)-5$. But now, it contradicts that a planar graph with $n+2$ (≥ 3) vertices has at most $3(n+2)-6$ edges. $\square$

Definition. Let $\gamma = (G, a_0, a_1, a_2, b_1, b_2)$ be a rooted graph with an ideal frame A, B w.r.t. a_0. Let $a'b', a''b'' \in E(G)$ with $a', a'' \in V(A)$ and $b', b'' \in V(B)$ all distinct. We say that $a'b', a''b''$ form a *cross* (w.r.t. A, B) if a_1, a', a'', a_2 occur on A in order, and b_1, b'', b', b_2 occur on B in order. We say that $a'b', a''b''$ are *parallel* if a_1, a', a'', a_2 occur on A in order, and b_1, b', b'', b_2 occur on B in order.

For $i=5,6,7$, let $e_i = a_ib_i \in E(G)$ with $a_i \in V(A)$ to $b_i \in V(B)$. We say that (e_5, e_6, e_7) is a *3-edge configuration* (w.r.t. A, B) if $b_6 \in B(b_5, b_7)$ and $a_1, a_2, a_6 \notin A[a_5, a_7]$.

For $i=3,4,5,6,7$, let $e_i = a_ib_i \in E(G)$ with $a_i \in V(A)$ and $b_i \in V(B)$. We say that

$(e_3, e_4, e_5, e_6, e_7)$ is a *5-edge configuration* (w.r.t. A, B) if

- (e_5, e_6, e_7) is a 3-edge configuration w.r.t. A, B,
- $A[a_5, a_7] \subseteq A(a_3, a_4)$, and
- $b_3, b_4 \in B(b_j, b_5) \cap B(b_j, b_7)$ for some $j \in [2]$.

Lemma 6.0.2 *Suppose γ is infeasible, and A', B' is a core a_0-frame in γ. Suppose A, B is an ideal frame w.r.t. a_0 in γ. Then there exists a 5-edge configuration w.r.t. A, B.*

Proof. (1) For any $i \in [2]$, G has a cross from $A - a_i$ to B.

For, suppose (1) fails. Without loss of generality, we assume G has no cross from $A - a_2$ to B. Now, we let $b' \in B[b_1, b_2]$, such that G has an edge e' from b' to $A[a_1, a_2)$, and subject to this, $B[b', b_2]$ is minimal.

We first see that G has an edge from a_2 to $B[b_1, b')$; otherwise, $(G, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

Now, we let $u_1, u_2 \in B[b_1, b')$, such that G has an edge from u_k to a_2 for each $k \in [2]$, and subject to this, $B[u_1, u_2]$ is maximal.

We claim that G has an edge e from $b \in B(u_1, u_2)$ to $a \in A[a_1, a_2)$. For otherwise, we can obtain a new graph H from G by splitting a_2 as s, s', such that H has no edge from $B[u_1, u_2]$ to s' and no edge from $B[b', b_2]$ to s, and (H, a_1, b_2, a_0, b_1) is planar, which contradicts (i) of Lemma 6.0.1.

We also see that $a \notin A(a_1, a_2)$. For otherwise, let $e^* = a_1 b^* \in E(G)$ with $b^* \neq b$. Since G has no cross from $A - a_2$ to B, then $b^* \in B(b_1, b)$. Now, $(e^*, u_1 a_2, e, u_2 a_2, e')$ is a 5-edge configuration, a contradiction.

So, $a = a_1$, and all edges from $B(u_1, u_2)$ to $A[a_1, a_2)$ are end in a_1. But now, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, contradicting (ii) of Lemma 6.0.1. □

We let $b'_1, b'_2 \in B[b_1, b_2]$, such that b_1, b'_1, b'_2, b_2 occur on B in order, G has an edge from b'_i to A for each $i \in [2]$, and subject to this, $B[b'_1, b'_2]$ is maximal.

(2) For some $i \in [2]$, G has no edge from b'_i to $A(a_1, a_2)$.

For, suppose G has an edge e'_i from b'_i to $A(a_1, a_2)$ for each $i \in [2]$. For each $k \in [2]$, since the degree of a_k in G is at least 4, then we may assume G has an edge e_k from a_k to $B(b'_1, b'_2)$. But now, e_1, e_2, e'_1, e'_2 form a doublecross, a contradiction. □

By symmetry, without loss of generality, we may assume that G has no edge from b'_1 to $A(a_1, a_2)$, and has an edge e_3 from b'_1 to a_1.

By (1), there exist $e_4 = a_4 b_4, e_5 = a_5 b_5 \in E(G)$ with $a_4, a_5 \in A(a_1, a_2]$ and $b_4, b_5 \in B[b'_1, b_2]$, such that e_4, e_5 form a cross, and b_1, b_4, b_5, b_2 occur on B in order. We further choose e_4, e_5 so that $B[b'_1, b_4] \cup A[a_1, a_5]$ is minimal, and subject to this, $B[b_5, b_2] \cup A[a_4, a_2]$ is minimal. By the choice of e_4, e_5, we may assume

(3) G has no edge from $B[b_1, b_4)$ to $A(a_5, a_2]$, no edge from $A(a_1, a_5)$ to $B(b_4, b_2]$, no edge from b_4 to $A(a_4, a_2]$, and no edge from a_5 to $B(b_5, b_2]$.

(4) G has no cross from $B[b_1, b_4]$ to $A[a_1, a_5]$ and no cross from $B[b_5, b_2]$ to $A[a_4, a_2]$.

For otherwise, such a cross together with e_4, e_5 forms a doublecross. □

(5) If G has an edge from $B(b_5, b_2]$ to $A(a_1, a_4)$, then G has no edge from $B(b_4, b_5)$ to $A(a_1, a_4) - a_5$.

For, suppose G has an edge e from $b \in B(b_5, b_2]$ to $a \in A(a_1, a_4)$ and an edge e' from $b' \in B(b_4, b_5)$ to $a' \in A(a_1, a_4) - a_5$. Now, by (3), we have $a \notin A(a_1, a_5]$, $a' \notin A(a_1, a_5)$, and so $a, a' \in A(a_5, a_4)$. But then, (e_3, e_4, e', e_5, e) is a 5-edge configuration. □

Let $b'_5 \in B(b_4, b_5]$, such that G has an edge e'_5 from a_5 to b'_5, and subject to this, $B[b'_5, b_2]$ is maximal.

(6) G has no edge from $B(b'_5, b_5)$ to $A - a_5$.

For, suppose G has an edge e from $B(b'_5, b_5)$ to $A - a_5$. Then $b_5 \neq b'_5$, and (e_3, e_4, e'_5, e, e_5) forms a 5-edge configuration, a contradiction. □

(7) $G - a_4b'_5$ has no cross from $B[b'_5, b_2]$ to $A(a_5, a_2]$.

For, suppose there exist $e' = a'b', e'' = a''b'' \in E(G)$ with $a', a'' \in A(a_5, a_2]$ and $b', b'' \in B[b'_5, b_2]$, such that e', e'' form a cross, a_1, a', a'', a_2 occur on A in order, and $e'' \neq a_4b'_5$. Then we may assume $a'' \notin A(a_4, a_2]$; otherwise, e_4, e'_5, e', e'' form a doublecross. So $a' \in A(a_5, a_4)$. Now, we see that $b'' = b'_5$; otherwise, $(e_3, e_4, e'_5, e'', e')$ is a 5-edge configuration. Since $e'' \neq a_4b'_5$, then $a'' \neq a_4$. So $a'' \in A(a_5, a_4)$. Now, let $e^* = a''b^* \in E(G)$ with $b^* \in B[b_1, b_2]$. Since the degree of a'' in G is at least 6, we may further let $b^* \notin \{b', b'', b_4\}$. First, we see that $b^* \notin B[b_1, b_4)$; otherwise, e^*, e', e_4, e'_5 form a doublecross. Next, $b^* \notin B(b_4, b'_5)$; otherwise, $(e_3, e_4, e^*, e'_5, e')$ is a 5-edge configuration. Moreover, $b^* \notin B(b'_5, b')$; otherwise, $(e_3, e_4, e'_5, e^*, e')$ is a 5-edge configuration. So we may assume $b^* \in B(b', b_2]$, but then (e_3, e_4, e'', e', e^*) is a 5-edge configuration, a contradiction. □

If $a_4 \neq a_2$, we let $b^*_1, b^*_2 \in B(b_4, b_2]$, such that b_1, b^*_1, b^*_2, b_2 occur on B in order, G has an edge e^*_i from $a^*_i \in A(a_4, a_2]$ to b^*_i for $i \in [2]$, and subject to this, $B[b^*_1, b^*_2]$ is maximal.

(8) If $a_4 \neq a_2$, then G has no edge from $B(b^*_1, b^*_2)$ to a_5.

For, suppose $a_4 \neq a_2$, and G has an edge e^*_5 from $b^*_5 \in B(b^*_1, b^*_2)$ to a_5. We see that $b^*_2 \neq b_2$. For otherwise, $b^*_2 = b_2$ and $a^*_2 \neq a_2$. By (3), G has no edge from a_2 to $B[b_1, b_4]$, and so G has an edge from a_2 to $B(b_4, b_2)$, which together with e_4, e^*_2, e^*_5 forms a doublecross. Then we shall show that we can obtain a new graph H from G by splitting a_5 or b^*_5 as s, s', such that (H, a_1, b_2, a_0, b_1) is planar.

We first claim that G has no edge from $B[b_1, b^*_1)$ to $A(a_4, a_2]$ and no cross from $B[b_1, b^*_1)$ to $A[a_1, a_2]$. In fact, we see that G has no edge from $B(b_4, b^*_1)$ to $A[a_1, a_2] - a_4$. For otherwise, let $e = ab \in E(G)$ with $b \in B(b_4, b^*_1)$ and $a \in A[a_1, a_2] - a_4$. Then by the definition of b^*_1, b^*_2, we have $a \notin A(a_4, a_2]$. Moreover, $a \neq a_1$ to avoid the doublecross e, e_4, e^*_5, e^*_1. But then $a \in A(a_1, a_4)$, and so $(e_3, e_4, e, e^*_1, e^*_5)$ is a 5-edge configuration, a contradiction. Now, combined with (3) and (4), we may assume G has no edge from $B[b_1, b^*_1)$ to $A(a_4, a_2]$ and no cross from $B[b_1, b^*_1)$ to $A[a_1, a_2]$.

We also claim that G has no edge from $B(b_2^*, b_2]$ to $A[a_1, a_2]$. For, suppose G has an edge e from $b \in B(b_2^*, b_2]$ to $a \in A[a_1, a_2]$. Then $a \neq a_1$; otherwise, $(e, e_2^*, e_5^*, e_1^*, e_4)$ is a 5-edge configuration. And $a \notin A(a_1, a_4)$; otherwise, $(e_3, e_4, e_5^*, e_2^*, e)$ is a 5-edge configuration. We may also assume $a \neq a_4$; otherwise, e_4, e_1^*, e_5^*, e form a doublecross. So $a \in A(a_4, a_2]$, but it contradicts the definition of b_1^*, b_2^*.

Moreover, we claim that $G - \{a_5, b_5^*\}$ has no edge e from $B[b_1^*, b_2^*]$ to $A[a_1, a_4]$, such that $e \neq a_4 b_1^*$. For, suppose there exists $e = ab \in E(G)$ with $e \neq a_4 b_1^*$, $a \in A[a_1, a_4] - a_5$, and $b \in B[b_1^*, b_2^*] - b_5^*$. Then we may assume $b \notin B(b_5^*, b_2^*]$; otherwise, if $a \in A[a_1, a_5)$, then $a = a_1$ by (3), and $(e_2^*, e, e_5^*, e_1^*, e_4)$ is a 5-edge configuration; if $a \in A(a_5, a_4]$, then e_4, e_5^*, e, e_1^* form a doublecross. So $b \in B[b_1^*, b_5^*)$. Now, if $a \in A[a_1, a_5)$, then e_4, e_5^*, e, e_1^* form a doublecross. So $a \in A(a_5, a_4]$. We may further assume $b = b_1^*$; or else, $b \in B(b_1^*, b_5^*)$, and $(e_2^*, e_5^*, e, e_1^*, e_4)$ is a 5-edge configuration. Since $e \neq a_4 b_1^*$, then $a \in A(a_5, a_4)$. Now, we let $e_0 = ab_0 \in E(G)$ with $b_0 \in B[b_1, b_2]$. Since the degree of a in G is at least 6, we may further let $b_0 \notin \{b_4, b_1^*, b_5^*\}$. By (3), $b_0 \notin B[b_1, b_4)$. Moreover, $b_0 \notin B(b_4, b_1^*)$; or else, $(e_3, e_4, e_0, e_1^*, e_5^*)$ is a 5-edge configuration. By $b = b_1^*$, we have $b_0 \notin B(b_1^*, b_2^*] - b_5^*$. So $b_0 \in B(b_2^*, b_2]$, but it contradicts that G has no edge from $B(b_2^*, b_2]$ to $A[a_1, a_2]$.

Finally, we claim that G has no cross from $A(a_4, a_2]$ to $B[b_1^*, b_5^*) \cup B(b_5^*, b_2^*]$. For, suppose there exist $e' = a'b', e'' = a''b'' \in E(G)$ with $a', a'' \in A(a_4, a_2]$ and $b', b'' \in B[b_1^*, b_5^*) \cup B(b_5^*, b_2^*]$, such that e', e'' form a cross, and a_1, a', a'', a_2 occur on A in order. Then $b' \in B[b_1^*, b_5^*)$ to avoid the doublecross e_4, e_5^*, e', e'', and so $b'' \in B[b_1^*, b_5^*)$. Moreover, $a_2^* \in A[a'', a_2]$ to avoid the doublecross e_4, e_5^*, e'', e_2^*. But now, $(e_2^*, e_5^*, e', e'', e_4)$ is a 5-edge configuration.

Now, we let $e' = a'b', e'' = a''b'' \in E(G)$ with $b' \in B[b_1^*, b_5^*)$, $b'' \in B(b_5^*, b_2^*]$, and $a', a'' \in A(a_4, a_2]$, such that $B[b', b'']$ is minimal.

We may assume G has an edge e_0 from b_5^* to $a_0 \in A[a_1, a') \cup A(a'', a_2]$ with $a_0 \neq a_5$. For otherwise, combined with (6) and our claims, we can obtain a new graph H from G by splitting a_5 as s, s', such that (H, a_1, b_2, a_0, b_1) is planar, which contradicts (i) of

Lemma 6.0.1.

To avoid the doublecross e_5^*, e_0, e'', e_4, we may further assume $a_0 \in A[a_1, a')$.

Now, we claim that G has no edge from a_5 to $B(b_4, b_2] - b_5^*$. For, suppose G has an edge e from a_5 to $b \in B(b_4, b_2] - b_5^*$. Assume $b \in B(b_4, b_5^*)$. Now, if $a_0 \in A(a_5, a')$, then e, e_0, e_4, e' form a doublecross. So $a_0 \in A[a_1, a_5)$. Now, let $e_6 = a_5b_6 \in E(G)$. Since the degree of a_5 in G is at least 6, then we may let $b_6 \notin \{b_4, b', b_5^*\}$. To avoid the doublecross e_6, e_0, e_4, e', $b_6 \notin B(b_5^*, b_2]$. Moreover, $b_6 \notin B(b', b_5^*)$; or else, $(e_2^*, e_0, e_6, e', e_4)$ is a 5-edge configuration. By (6), $b_6 \notin B(b_4, b')$. So $b_6 \in B[b_1, b_4)$. But then $(e_2^*, e_0, e, e_4, e_6)$ is a 5-edge configuration. So we may assume $b \in B(b_5^*, b_2]$. By (6), $b \notin B(b_2^*, b_2]$. Now, if $a_0 \in A[a_1, a_5)$, then e_0, e, e_4, e' form a doublecross; if $a_0 \in A(a_5, a')$, then (e_2^*, e, e_0, e', e_4) is a 5-edge configuration, a contradiction.

Hence, by our claims, we can obtain a new graph H from G by splitting b_5^* as s, s', such that (H, a_1, b_2, a_0, b_1) is planar, which still contradicts (i) of Lemma 6.0.1. □

For each a_j, we let $u_1^j, u_2^j \in B[b_1, b_2]$, such that b_1, u_1^j, u_2^j, b_2 occur on B in order, G has an edge f_i^j from a_j to u_i^j for $i \in [2]$, and subject to this, $B[u_1^j, u_2^j]$ is maximal.

(9) If $a_4 \neq a_2$, then G has an edge from a_2 to $B(b_5, b_2]$.

For, suppose $a_4 \neq a_2$ and G has no edge from a_2 to $B(b_5, b_2]$. Since the degree of a_2 in G is at least 4, then, combined with $a_4 \neq a_2$ and the choice of e_4, e_5, we have $u_1^2, u_2^2 \in B(b_4, b_5]$, $u_1^2 \neq u_2^2$, and G has an edge f_2 from a_2 to $B(u_1^2, u_2^2)$. Then we shall show that (G, a_1, b_2, a_0, b_1) is planar.

We claim that G has no edge from $B(u_1^2, u_2^2)$ to $A[a_1, a_2)$. For, suppose G has an edge e from $b \in B(u_1^2, u_2^2)$ to $a \in A[a_1, a_2)$. First, $a \notin A[a_1, a_5)$; otherwise, e, e_4, e_5, f_1^2 form a doublecross. By (8), $a \neq a_5$. So $a \in A(a_5, a_2)$. Now, we may assume $b_5 = b_2$; otherwise, $(e_5, f_2^2, e, f_1^2, e_4)$ is a 5-edge configuration, a contradiction. Since $b_5 = b_2$, then $u_2^2 \neq b_5$. But now, $(e_3, f_1^2, e, f_2^2, e_5)$ is a 5-edge configuration, a contradiction.

We also claim that G has no cross from $A[a_1, a_2)$ to $B[b_1, u_1^2]$. For, suppose there exist $e' = a'b', e'' = a''b'' \in E(G)$ with $a', a'' \in A[a_1, a_2)$ and $b', b'' \in B[b_1, u_1^2]$, such that

e', e'' form a cross, and a_1, a', a'', a_2 occur on A in order. We see that $b'' \in B[b_4, u_1^2]$. For otherwise, $b'' \in B[b_1, b_4)$, and by the choice of e_4, e_5, $a'' \in A[a_1, a_5]$ and $a' = a_1$. But now e', e'', e_4, e_5 form a doublecross, a contradiction. Moreover, $a' = a_1$; otherwise, (e_3, e'', e', f_2, e_5) is a 5-edge configuration. But then, e', f_2, e_4, e_5 form a doublecross.

Finally, we claim that G has no parallel edges from $A[a_1, a_2)$ to $B[u_2^2, b_2]$. For, suppose there exist $e' = a'b', e'' = a''b'' \in E(G)$ with $a', a'' \in A[a_1, a_2)$ and $b', b'' \in B[u_2^2, b_2]$, such that e', e'' are parallel, and a_1, a', a'', a_2 occur on A in order. We see that $a' \in A[a_4, a_2)$; otherwise, e', e'', e_4, f_1^2 form a doublecross. Moreover, $b'' \in B[u_2^2, b_5]$; otherwise, e_5, e'', e_4, f_1^2 form a doublecross. We may assume $b_5 = b_2$; otherwise, $(e_5, e'', e', f_1^2, e_4)$ is a 5-edge configuration. So $u_2^2 \neq b_5$. Now, let $e = a''b \in E(G)$ with $b \notin \{b', b_5\}$. Then $b \notin B[b_1, u_1^2]$ to avoid the doublecross e, e'', f_2^2, e'. Moreover, $b \notin B[u_2^2, b')$; otherwise, (e_3, f_1^2, e, e', e'') is a 5-edge configuration. Since G has no edge from $B(u_1^2, u_2^2)$ to $A[a_1, a_2)$, then $b \in B(b', b_5)$. But now, (e_3, f_1^2, e', e, e_5) is a 5-edge configuration.

Hence, by our claims, (G, a_1, b_2, a_0, b_1) is planar, contradicting (i) of Lemma 6.0.1. □

(10) G has no edge from $B(b_5, b_2]$ to $A(a_1, a_4)$.

For, suppose G has an edge e from $b \in B(b_5, b_2]$ to $a \in A(a_1, a_4)$. We choose e so that $B[b, b_2]$ is minimal. By (3), $a \in A(a_5, a_4)$. By (5), G has no edge from $B(b_4, b_5)$ to $A(a_1, a_4) - a_5$. Moreover, since the degree of a in G is at least 6, then we let $e_0 = ab_0$ with $b_0 \in B[b_1, b_2]$ and $b_0 \notin \{b_4, b_5, b\}$. Now, by (3) and (5), and by the definition of b, we have $b_0 \in B(b_5, b)$.

G has no edge from $A(a_4, a_2]$ to $B[b_1, b)$. For, suppose there exists $e' = a'b' \in E(G)$ with $a' \in A(a_4, a_2]$ and $b' \in B[b_1, b)$. Then by (3), $b' \notin B[b_1, b_4]$. So $b' \in B(b_4, b)$. But then, e, e', e_4, e_5 form a doublecross.

G has no edge from b_4 to $A(a_5, a_4)$ or no edge from a_4 to $B(b_4, b)$; otherwise, such two edges together with e_5, e form a doublecross, a contradiction.

Now, we see that G has an edge e' from a_1 to $b' \in B(b_4, b_2]$; otherwise, since G has no edge from b_4 to $A(a_5, a_4)$ or no edge from a_4 to $B(b_4, b)$, then combined with (3), (4), (6), and (7), we can obtain a new graph H from G by splitting a_4 or b_4 as s, s', such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

We also see that G has no edge from a_1 to $B(b_5', b)$; otherwise, such an edge together with e_3, e_4, e_5', e forms a 5-edge configuration, a contradiction.

Hence, $b' \in B(b_4, b_5'] \cup B[b, b_2]$. We further choose e' so that $B[b', b_2]$ is maximal. Moreover, we let $e'' = a_1b'' \in E(G)$ with $b'' \in B(b_4, b_5'] \cup B[b, b_2]$ so that $B[b'', b_2]$ is minimal.

Now, assume $b'' \in B(b_4, b_5']$. Then by the choice of e'', G has no edge from a_1 to $B[b, b_2]$. Moreover, G has no edge from $B[b_1, b_4)$ to $A(a_1, a_2]$; otherwise, by (3), such an edge must end in $A(a_1, a_5]$, which together with e', e_4, e_5 forms a doublecross. Hence, G has an edge e_6 from a_4 to $b_6 \in B(b_4, b_5)$; or else, we can obtain a new graph H from G by splitting b_4 as s, s', such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1. Now, G has no edge from b_4 to $A(a_1, a_4)$; or else, such an edge together with e_5, e', e_6 forms a doublecross. So we may assume $a_2 \neq a_4$; otherwise, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, a contradiction to (ii) of Lemma 6.0.1. Then $u_2^2 \in B[b, b_2]$ (by (7) and (9)). Moreover, $b_6 \notin B(b', b_5]$; otherwise, (f_2^2, e, e_6, e', e_4) is a 5-edge configuration. So G has no edge from a_4 to $B(b', b_5]$. Therefore, we can obtain a new graph H from G by splitting a_4 as s, s', such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

So we may assume $b'' \in B[b, b_2]$. Now, $a_2 = a_4$; otherwise, $u_2^2 \in B[b, b_2]$ (by (7) and (9)) and $(f_2^2, e'', e_0, e_5, e_4)$ is a 5-edge configuration.

We also claim that G has an edge e_6 from $a_6 \in A(a_1, a_2)$ to $b_6 \in B[b_1, b_4]$; otherwise, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, a contradiction to (ii) of Lemma 6.0.1.

Then $b_6 \notin B[b_1, b_4)$; otherwise, $a_6 \in A(a_1, a_5]$, and (e, e'', e_5, e_4, e_6) is a 5-edge configuration. Hence, $b_6 = b_4$, and G has no edge from a_5 to $B[b_1, b_4)$, which further implies $b_5' \neq b_5$ (as the degree of a_5 in G is at least 6).

Now, we may assume $u_2^2 \notin B[b, b_2]$. For, suppose not. Then G has no edge from $\{a_1, a_2\}$ to $B(b_4, b_5)$; otherwise, such an edge together with f_2^2, e'', e_5, e_6 forms a 5-edge configuration. Moreover, $a_6 \notin A(a_5, a_2)$; otherwise, $(f_2^2, e'', e_0, e_5, e_6)$ is a 5-edge configuration. But now, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, a contradiction to (ii) of Lemma 6.0.1.

Since $u_2^2 \notin B[b, b_2]$, then G has no edge from a_2 to $B[b, b_2]$. By (7), G has no edge from a_2 to $B(b_5', b)$. By (3), G has no edge from a_2 to $B[b_1, b_4)$. Since the degree of a_2 in G is at least 4, then G has an edge e_2' from a_2 to $B(b_4, b_5')$. Now, $a_6 \notin A(a_5, a_2)$; otherwise, e_6, e_5, e, e_2' form a doublecross. Moreover, $b' \notin B(b_4, b)$ to avoid the doublecross e', e_2', e_6, e. Hence, combined with (6), we can obtain a new graph H from G by splitting a_2 as s, s', such that (H, a_1, b_2, a_0, b_1) is planar, a contradiction to (i) of Lemma 6.0.1. □

Now, by (3), (8), (9), and (10), we have

(11) G has no edge from $A(a_1, a_5) \cup A(a_4, a_2]$ to $B(b_4, b_5)$ and no edge from $B[b_1, b_4) \cup B(b_5, b_2]$ to $A(a_5, a_4)$.

(12) There do not exist $e' = a'b', e'' = a''b'' \in E(G)$ with $a', a'' \in A[a_5, a_4]$ and $b', b'' \in B[b_4, b_5]$, such that e', e'' are parallel, a_1, a', a'', a_2 occur on A in order, and $e' \neq a_5b_4, e'' \neq a_4b_5$.

For, suppose such e', e'' exist. Then $b' = b_4$ or $b'' = b_5$; otherwise, (e_3, e_4, e', e'', e_5) is a 5-edge configuration, a contradiction.

We may further assume $b' = b_4$. For otherwise, $b'' = b_5$ and $a'' \neq a_4$. Now, let $e = a''b \in E(G)$ with $b \in B[b_1, b_2]$. Since the degree of a'' in G is at least 6, we may further let $b \notin \{b_4, b', b_5\}$. By (11), $b \notin B[b_1, b_4) \cup B(b_5, b_2]$. Moreover, $b \notin B(b_4, b')$; otherwise, (e_3, e_4, e, e', e'') is a 5-edge configuration. So $b \in B(b', b_5)$. But then (e_3, e_4, e', e, e_5) is a 5-edge configuration.

By $b' = b_4$, we have that $G - a_4b_5$ has no parallel edges from $B(b_4, b_5]$ to $A[a_5, a_4]$.

Now, since $e'' \neq a_4b_5$ and the degree of a'' in G is at least 6, then combined with (11), we may choose $b'' \neq b_5$, and so $b'' \in B(b_4, b_5)$.

Since $e' \neq a_5b_4$, then $a' \in A(a_5, a_4)$. Moreover, since the degree of a' in G is at least 6, we let $e = a'b \in E(G)$ with $b \in B[b_1, b_2]$ and $b \notin \{b_4, b'', b_5\}$. By (11), $b \notin B[b_1, b_4) \cup B(b_5, b_2]$. And $b \notin B(b_4, b'')$; otherwise, (e_3, e_4, e, e'', e_5) is a 5-edge configuration. So $b \in B(b'', b_5)$.

Now, G has no edge from a_2 to $B[b_5, b_2]$; otherwise, (f_2^2, e_5, e, e'', e') is a 5-edge configuration. Hence, $a_4 = a_2$ (by (9)), and so G has no edge from a_4 to $B[b_5, b_2]$. Moreover, G has no edge from a_1 to $B(b_4, b_5)$; otherwise, such an edge together with e', e_5, e'' forms a doublecross.

Hence, since $G - a_4b_5$ has no parallel edges from $B(b_4, b_5]$ to $A[a_5, a_4]$, then, combined with (3), (4) and (11), any two edges from $B[b_1, b_4]$ to A do not form a cross, and any two edges from $B(b_4, b_2]$ to A are not parallel, which further implies that (G, a_1, b_2, a_0, b_1) is planar, a contradiction to (i) of Lemma 6.0.1. □

(13) G has an edge e_0 from a_1 to $b_0 \in B(b_4, b_2]$.

For, suppose G has no edge from a_1 to $B(b_4, b_2]$. Then by (3), (4), (11), and (12), we can obtain a new graph H from G by splitting a_5, a_4 as s, s' and t, t', respectively, such that $(H, a_0, b_1, a_1, s, t, s', t', a_2, b_2)$ is planar, a contradiction to (iii) of Lemma 6.0.1. □

We choose e_0 so that $B[b_0, b_2]$ is maximal. Moreover, we let $e_0' = a_1b_0' \in E(G)$ with $b_0' \in B(b_4, b_2]$ so that $B[b_0', b_2]$ is minimal.

(14) $a_4 \neq a_2$.

For, suppose $a_4 = a_2$. We first claim that $A(a_5, a_2) \neq \emptyset$.

For otherwise, $A(a_5, a_2) = \emptyset$. Now, we may assume G has an edge e from $b \in B[b_1, b_4]$ to $a \in A(a_1, a_5]$; or else, combined with (3), (4) and (6), $(G - a_1, a_2, b_2, a_0, b_1)$ is planar and $(G - a_2, a_1, b_2, a_0, b_1)$ is planar, a contradiction to (ii) of Lemma 6.0.1.

Moreover, G has no edge from a_2 to $B(b_4, b_5)$. For, suppose G has an edge e' from a_2 to $b' \in B(b_4, b_5)$. Then G has no edge from a_1 to $B(b_4, b_5)$; or else, such an edge together with e, e', e_5 forms a doublecross. So $b_0 \in B[b_5, b_2]$. Now, G has no edge from a_2 to $B(b'_5, b_2]$; otherwise, such an edge together with e_0, e'_5, e', e forms a 5-edge configuration. So, combined with (3), (4) and (6), (G, a_1, b_2, a_0, b_1) is planar, a contradiction to (i) of Lemma 6.0.1.

Now, since G has no edge from a_2 to $B(b_4, b_5)$, then G has an edge from a_2 to $B(b_5, b_2]$ (by the degree of a_2 in G), and so $u_2^2 \in B(b_5, b_2]$.

Assume $b_0 \in B(b_4, b_5)$. Then $b \notin B[b_1, b_4)$ to avoid the doublecross e_0, e, e_4, e_5. Now, by the degree of a_5 in G, $b'_5 \neq b_5$, and G has an edge e''_5 from a_5 to $b''_5 \in B(b'_5, b_5)$. By (6), $b_0 \in B(b_4, b'_5]$. So G has no edge from a_1 to $B[b_5, b_2]$; otherwise, such an edge together with f_2^2, e''_5, e_0, e forms a 5-edge configuration. Hence, combined with (3), (4) and (6), we can obtain a new graph H from G by splitting b_4 as s, s', such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

Therefore, we may assume G has no edge from a_1 to $B(b_4, b_5)$. Moreover, G has an edge from $B[b_1, b_4)$ to $A(a_1, a_5]$; otherwise, $(G-a_1, a_2, b_2, a_0, b_1)$ and $(G-a_2, a_1, b_2, a_0, b_1)$ are planar, a contradiction to (ii) of Lemma 6.0.1. Hence, we may choose e so that $b \in B[b_1, b_4)$. Then $b'_0 \notin B(b_5, b_2]$ and $b'_5 = b_5$; otherwise, $(f_2^2, e'_0, e'_5, e_4, e)$ is a 5-edge configuration. Hence, $b_0 = b'_0 = b_5$, and we can obtain a new graph H from G by splitting b_5 as s, s', such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

Thus, our claim that $A(a_5, a_2) \neq \emptyset$ holds, and there exists $a_6 \in A(a_5, a_2)$. Since the degree of a_6 in G is at least 6 and G has no edge from a_6 to $B[b_1, b_4) \cup B(b_5, b_2]$ (by (11)), we may let $b'_6, b''_6 \in B(b_4, b_5)$ with $b'_6 \neq b''_6$, such that b_1, b'_6, b''_6, b_2 occur on B in order, G has an edge e'_6 from a_6 to b'_6 and an edge e''_6 from a_6 to b''_6, and subject to this, $B[b'_6, b''_6]$ is maximal.

We now claim that G has no edge from $B[b_1, b_4]$ to $A(a_1, a_5]$. For, suppose G has an edge e'' from $b'' \in B[b_1, b_4]$ to $a'' \in A(a_1, a_5]$. Then $b_0 \in B(b_4, b'_6]$ to avoid the doublecross e_0, e'', e_5, e''_6. Moreover, $b_0 \notin B(b'_6, b_5)$; otherwise, $(e_3, e_4, e'_6, e_0, e_5)$ is a 5-edge configuration. Hence, $b_0 \in B[b_5, b_2]$ and G has no edge from a_1 to $B(b_4, b_5)$. We also see that G has no edge from a_1 to $B(b_5, b_2]$ or no edge from a_2 to $B(b_5, b_2]$; otherwise, such two edges together with e_5, e'_6, e'' form a 5-edge configuration. But then, combined with (3), (4), (11), and (12), we can obtain a new graph H from G by splitting a_2 as s, s', such that (H, a_1, b_2, a_0, b_1) is planar, a contradiction to (i) of Lemma 6.0.1.

Thus, G has no edge from $B[b_1, b_4]$ to $A(a_1, a_5]$, and so by (11) and (12), $(G - a_1, a_2, b_2, a_0, b_1)$ is planar. Now, by (ii) of Lemma 6.0.1, $(G - a_2, a_1, b_2, a_0, b_1)$ is not planar, and so we may assume G has an edge e from a_1 to $b \in B(b_4, b_5)$ and an edge e' from $b' \in B[b_1, b)$ to $a' \in A(a_1, a_2)$. And $b \notin B(b'_6, b_5)$ to avoid 5-edge configuration (e_3, e_4, e'_6, e, e_5). Moreover, we may assume G has no edge from a_2 to $B(b_4, b_5)$; otherwise, such an edge together with e, e', e_5 forms a doublecross. So, by the degree of a_2, $u_2^2 \in B[b_5, b_2]$. But now, $(f_2^2, e_5, e''_6, e, e')$ is a 5-edge configuration. □

Now, by (9) and (14), G has an edge from a_2 to $B(b_5, b_2]$, and so $u_2^2 \in B(b_5, b_2]$. By (3), (11) and (14), G has no edge from a_2 to $B[b_1, b_5)$, and so $u_1^2 \in B[b_5, b_2]$.

(15) $b_0 \in B(b_4, b_5)$.

For otherwise, $b_0 \in B[b_5, b_2]$. Now, we see that $b'_0 \neq b_5$; otherwise, $b_0 = b'_0 = b_5$, and by (3), (4), (11), (12), and (14), we can obtain a new graph H from G by splitting a_4 as s, s', such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

Then G has no edge from $B[b_1, b_4)$ to $A(a_1, a_5]$; or else, such an edge together with f_2^2, e'_0, e_5, e_4 forms a 5-edge configuration. Hence, $A(a_1, a_5) = \emptyset$, and by the degree of a_5 in G, $b'_5 \neq b_5$. Now, G has no edge from $B[b_5, b'_0)$ to $A[a_4, a_2)$; otherwise, such an edge together with f_2^2, e'_0, e'_5, e_4 forms a 5-edge configuration.

Moreover, if $b_4a_5 \in E(G)$, then G has no edge from $B(b_4, b_5)$ to $A(a_5, a_2]$; otherwise, such an edge together with f_2^2, e'_0, e_5, a_5b_4 forms a 5-edge configuration.

We may assume $u_1^2 \in B[b_5, b'_0)$. For otherwise, G has no edge from $B[b_5, b'_0)$ to $A[a_4, a_2]$. Now, by (3), (4), (11), (12), and our previous statements, we can obtain a new graph H from G by splitting a_1, a_4 as s, s' and t, t', respectively, such that $a_1 := s$ in H, and $(H, a_0, b_1, a_1, s, t, s', t', a_2, b_2)$ is planar, a contradiction to (iii) of Lemma 6.0.1.

Now, by $u_1^2 \in B[b_5, b_0')$, G has no edge from $B[b_5, b_2]$ to $A[a_4, a_2)$, and so, by (3), (4), (11), (12), and our previous statements, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, a contradiction to (ii) of Lemma 6.0.1. □

Now, we see that $A(a_5, a_4) = \emptyset$. For otherwise, there exists $a \in A(a_5, a_4)$. Since the degree of a in G is at least 6, then combined with (11), G has an edge e from a to $b \in B[b_4, b_5]$ with $b \notin \{b_4, b_5, b_0\}$. Now, if $b \in B(b_4, b_0)$, then (e_3, e_4, e, e_0, e_5) is a 5-edge configuration; if $b \in B(b_0, b_5)$, then $(f_2^2, e_5, e, e_0, e_4)$ is a 5-edge configuration.

We may assume G has no edge from $A(a_1, a_5]$ to $B[b_1, b_4)$; otherwise, such an edge together with e_0, e_4, e_5 forms a doublecross.

Then we claim that G has no edge from $B(b_0, b_0')$ to $A(a_1, a_2]$. For, suppose G has an edge e from $b \in B(b_0, b_0')$ to $a \in A(a_1, a_2]$. Then $b_0' \notin B(b_5, b_2]$; otherwise, $(f_2^2, e_0', e_5, e_0, e_4)$ is a 5-edge configuration. So $b_0' \in B(b_4, b_5]$. Hence, $b \in B(b_4, b_5)$, and by (11), $a \in A[a_5, a_4]$. But then, $(f_2^2, e_0', e, e_0, e_4)$ is a 5-edge configuration.

Now, if G has no edge from a_4 to $B(b_4, b_5)$, then, combined with (3), (4), (6), (11), and our previous statements, we can obtain a new graph H from G by splitting b_4 as s, s', such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

So we may assume G has an edge e from a_4 to $b \in B(b_4, b_5)$. Then $b \notin B(b_0, b_5)$; otherwise, $(f_2^2, e_5, e, e_0, e_4)$ is a 5-edge configuration. Moreover, G has no edge from b_4 to a_5, since, otherwise, such an edge together with e_5, e_0, e forms a doublecross. But now, combined with (3), (4), (6), (11), and our previous statements, we can obtain a new graph H from G by splitting a_4 as s, s', such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1. □

Lemma 6.0.3 *Suppose γ is infeasible, and A', B' is a core a_0-frame in γ. Suppose $\mathcal{P} = (e_3, e_4, e_5, e_6, e_7)$ is a 5-edge configuration w.r.t. an ideal frame A, B in γ with a_1, a_3, a_4, a_2*

on A in order and $b_1, b_3, b_4, b_5, b_6, b_7, b_2$ on B in order. Let $G_0 := G - A$, where (G_0, a_0, b_1, B, b_2) is planar.

Then G_0 has a separation (G_1, G_2) such that $b'_1, b'_2 \in V(G_1) \cap V(G_2)$, $|V(G_1) \cap V(G_2)| \leq 3$, $\{a_0, b_1, b_2\} \subseteq V(G_1)$, $B[b'_1, b'_2] \subseteq G_2$, $|V(G_1 - G_2)| \geq 1$, and one of the following holds:

(i) $V(G_1) \cap V(G_2) = \{a'_0, b'_1, b'_2\}$, $b'_1 \in B[b_3, b_4]$, $b'_2 \in B[b_7, b_2]$, *and G_0 has a path from a_0 to $B(b'_1, b'_2)$ through a'_0 and internally disjoint from B.*

(ii) $V(G_1) \cap V(G_2) = \{b'_1, b'_2\}$, $b'_1 \in B[b_3, b_4]$, *and* $b'_2 \in B[b_7, b_2]$.

(iii) $V(G_1) \cap V(G_2) = \{b'_1, b'_2\}$, $b'_1 \in B[b_3, b_4]$, *and* $b'_2 \in B[b_6, b_7)$.

(iv) $V(G_1) \cap V(G_2) = \{b'_1, b'_2\}$, $b'_1 \in B(b_4, b_5]$, *and* $b'_2 \in B[b_7, b_2]$.

Proof. Since otherwise, $G_0 - (B[b_3, b_4] \cup B[b_7, b_2])$ contains disjoint paths B_1, A_0 from b_1, a_0 to b_5, b_6, respectively. Now $(A - A[a_5, a_7]) \cup e_3 \cup B[b_3, b_4] \cup e_4 \cup e_6 \cup A_0$ and $B_1 \cup e_5 \cup A[a_5, a_7] \cup e_7 \cup B[b_7, b_2]$ show that γ is feasible, a contradiction. $\square$

Definition. Let $\gamma = (G, a_0, a_1, a_2, b_1, b_2)$ be a rooted graph. Suppose A, B is an ideal frame w.r.t. a_0 in γ, and $(e_3, e_4, e_5, e_6, e_7)$ is a 5-edge configuration w.r.t. A, B, where $e_i = a_i b_i \in E(G)$ with $a_i \in V(A)$ and $b_i \in V(B)$ for $i = 3, 4, 5, 6, 7$.

For notational convenience, we further assume a_1, a_3, a_4, a_2 occur on A in order, and $b_1, b_3, b_4, b_5, b_6, b_7, b_2$ occur on B in order. Then we say that $(e_3, e_4, e_5, e_6, e_7)$ is an *ideal* frame if the following requirements are satisfied in the order listed:

- $B[b_4, b_7]$ is maximal,
- $B[b_6, b_7]$ is minimal,
- $B[b_4, b_5]$ is minimal,
- $A[a_5, a_7]$ is minimal,

- $A[a_3, a_4]$ is maximal,
- $B[b_1, b_3]$ is minimal, and
- $A[a_6, a_5] \cap A[a_6, a_7]$ is maximal.

Lemma 6.0.4 *Suppose γ is infeasible, and A', B' is a core a_0-frame in γ. Suppose $\mathcal{P} = (e_3, e_4, e_5, e_6, e_7)$ is a 5-edge configuration w.r.t. an ideal frame A, B in γ with a_1, a_3, a_4, a_2 on A in order and $b_1, b_3, b_4, b_5, b_6, b_7, b_2$ on B in order. Let $G_0 := G - A$, where (G_0, a_0, b_1, B, b_2) is planar.*

Assume $a_7 \in A[a_1, a_5], a_6 \in A(a_5, a_2]$, G has no edge from $B(b_4, b_5]$ to $A[a_1, a_5)$, and G has no edge from $B[b_7, b_2]$ to $A(a_5, a_2]$. Then there does not exist a separation (G_1, G_2) in G_0 such that $V(G_1 \cap G_2) = \{b_1^*, b_2^*\}$ with $b_1^* \in B[b_1, b_4]$ and $b_2^* \in B[b_6, b_2]$, $\{a_0, b_1, b_2\} \subseteq V(G_1)$, $V(B[b_1^*, b_2^*]) \subseteq V(G_2)$, and $|V(G_1 - G_2)| \geq 1$.

Proof. We choose (G_1, G_2) so that $B[b_1^*, b_2^*]$ is maximized. To avoid forming a doublecross with e_5, e_6, we may assume

(1) G has no parallel edges from $B[b_6, b_2]$ to $A[a_1, a_5]$.

(2) If $ab \in E(G)$ with $a \in A(a_5, a_2]$ and $b \in V(B)$ then $b \in B[b_4, b_6]$.

Suppose $e = ab \in E(G)$ with $a \in A(a_5, a_2]$ and $b \in V(B) - V(B[b_4, b_6])$. We may assume $b \in B[b_1, b_4)$. For, if $b \in B(b_6, b_2]$, then since G has no edge from $B[b_7, b_2]$ to $A(a_5, a_2]$, $b \in B(b_6, b_7)$. Now (e_3, e_4, e_5, e, e_7) contradicts the choice of $\mathcal{P}$.

Suppose $a \in A(a_5, a_4)$. Then $b \in B[b_3, b_4)$ to avoid the doublecross e, e_3, e_4, e_5. But then $b_3 \neq b_4$, and (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$.

Thus $a \in A[a_4, a_2]$. Then $b = b_1$ as, otherwise, (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. Hence, $a \neq a_2$; and let $e' = a_2 b' \in E(G)$ with $b' \in V(B) - \{b_1, b_2\}$. Then $b' \in B[b_7, b_2)$ to avoid the doublecross e, e', e_3, e_7. But it contradicts that G has no edge from $B[b_7, b_2]$ to $A(a_5, a_2]$. $\square$

Let $e_9 = a_9b_9$ with $a_9 \in A[a_1, a_5]$ and $b_9 \in B(b_1^*, b_2^*)$, and choose e_9 so that $A[a_1, a_9]$ is minimal.

Since G^* is 6-connected, then

(3) There exists $e^* = a^*b^* \in E(G)$ with $a^* \in A(a_9, a_2]$ and $b^* \in B - B[b_1^*, b_2^*]$.

By (2), we have $a^* \notin A(a_5, a_2]$; so $a^* \in A(a_9, a_5]$ and $a_9 \in A[a_1, a_5)$. Moreover,

(4) $b_9 \in B(b_1^*, b_4] \cup B[b_6, b_2^*)$.

For otherwise, $b_9 \in B(b_5, b_6)$ by $a_9 \in A[a_1, a_5)$ and the assumption that G has no edge from $B(b_4, b_5]$ to $A[a_1, a_5)$. Now, $a_9 \in A[a_7, a_5)$ to avoid the doublecross e_5, e_6, e_7, e_9. By $a^* \in A(a_9, a_5]$, $b^* \in B[b_1, b_1^*)$ to avoid the doublecross e_9, e^*, e_5, e_6, and $b^* \notin B[b_1, b_3)$ to avoid the doublecross e_3, e^*, e_6, e_7. Hence, $b_3, b^* \in B(b_1, b_4)$, and $(e_3, e^*, e_9, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. $\square$

Case 1. $b_9 \in B[b_6, b_2^*)$.

Then $b^* \in B[b_1, b_1^*)$ to avoid the doublecross e_9, e^*, e_5, e_6.

We claim that G has no edge from $B(b_1^*, b_4]$ to $A[a_1, a_5)$. For suppose $e = ab \in E(G)$ with $a \in A[a_1, a_5)$ and $b \in B[b_1^*, b_4]$. Note that b_1^* and b_2^* are feet of some connector J, and $B[b_1^*, b_2^*] \subseteq J$. Let u_1, u_2 denote the extreme hands for J. Note that e^* is from $A(x_1, x_2)$ to $B[b_1, b_1^*)$; so we know $(J - b_1^*, u_1, A(u_1, u_2), u_2, b_2^*)$ is planar by Lemma 3.0.4. But this cannot be the case because of e, e_4, e_5.

Because of (G_1, G_2), G has a separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{b_1', b_2'\}$ with $b_1^*, b_1', b_4, b_6, b_2', b_2^*$ on B in order, $B[b_1', b_2'] \subseteq G_1'$, and $\{a_0, b_1', b_2'\} \subseteq V(G_2')$. We choose (G_1', G_2') such that $B[b_6, b_2']$ is minimal and, subject to this, $B[b_1^*, b_1']$ is minimal.

Let $e_9' = a_9'b_9' \in E(G)$ with $a_9' \in A[a_1, a_5]$ and $b_9' \in B(b_1', b_2')$, and choose e_9' so that $A[a_1, a_9']$ is minimal. We may assume that there exists $e' = a'b' \in E(G)$ with $a' \in A(a_9', a_2]$ and $b' \in B - B[b_1', b_2']$ (as G^* is 6-connected).

Then $b_9' \in B[b_6, b_2')$ (since G has no edge from $B(b_1^*, b_4]$ to $A[a_1, a_5)$) and $b' \in B[b_1, b_1^*] - b_1'$ (to avoid doublecross with e_5, e_6, e_9', e'). So $(e_9', e_6, e_5, e_4, e')$ is a 5-edge configuration. By Lemma 2.0.9 and 6.0.3, G_0 has a cut that contradicts the choice of (G_1, G_2) or (G_1', G_2').

Case 2. $b_9 \in B(b_1^*, b_4]$.

Then $b^* \in B(b_2^*, b_2]$ to avoid the doublecross e_9, e^*, e_4, e_5.

We claim that G has no edge from $B[b_6, b_2^*)$ to $A[a_1, a_5)$. For suppose $e = ab \in E(G)$ with $a \in A[a_1, a_5)$ and $b \in B[b_6.b_2^*)$. Note that b_1^* and b_2^* are feet of some connector J, and $B[b_1^*, b_2^*] \subseteq J$. Let u_1, u_2 denote the extreme hands for J. Note that e^* is from $A(u_1, u_2)$ to $B(b_2^*, b_2]$; so we know $(J - b_2^*, u_1, A(u_1, u_2), u_2, b_1^*)$ is planar by Lemma 3.0.4. But this cannot be the case because of e, e_5, e_6.

Because of (G_1, G_2), G has a separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{b_1', b_2'\}$ with $b_1^*, b_1', b_4, b_6, b_2', b_2^*$ on B in order, $B[b_1', b_2'] \subseteq G_1'$, and $\{a_0, b_1', b_2'\} \subseteq V(G_2')$. We choose (G_1', G_2') such that $B[b_1', b_4]$ is minimal and, subject to this, $B[b_2', b_2^*]$ is minimal.

Let $e_9' = a_9'b_9' \in E(G)$ with $a_9' \in A[a_1, a_5]$ and $b_9' \in B(b_1', b_2')$, and choose e_9' so that $A[a_1, a_9']$ is minimal. We may assume that there exists $e' = a'b' \in E(G)$ with $a' \in A(a_9', a_2]$ and $b' \in B - B[b_1', b_2']$ (as G^* is 6-connected).

Then $b_9' \in B(b_1', b_4]$ (since G has no edge from $B[b_6, b_2^*]$ to $A[a_1, a_5)$) and $b' \in B[b_2^*, b_2] - b_2'$ (to avoid doublecross e', e_9', e_4, e_5). So $(e_9', e_4, e_5, e_6, e')$ is a 5-edge configuration. By Lemma 2.0.9 and 6.0.3, G_0 has a separation that contradicts choice of (G_1, G_2) or (G_1', G_2').

□

Lemma 6.0.5 *Suppose γ is infeasible, and A', B' is a core a_0-frame in γ. Suppose $\mathcal{P} = (e_3, e_4, e_5, e_6, e_7)$ is a 5-edge configuration w.r.t. an ideal frame A, B in γ with a_1, a_3, a_4, a_2 on A in order and $b_1, b_3, b_4, b_5, b_6, b_7, b_2$ on B in order. Let $G_0 := G - A$, where (G_0, a_0, b_1, B, b_2) is planar.*

Then G_0 has a separation (G_1, G_2) such that $b_1', b_2' \in V(G_1) \cap V(G_2)$, $|V(G_1) \cap V(G_2)| \leq 3$, $|V(G_1 - G_2)| \geq 1$, $\{a_0, a_1, a_2\} \subseteq V(G_1)$, $B[b_1', b_2'] \subseteq G_2$, and one of the following holds:

(i) $V(G_1) \cap V(G_2) = \{a'_0, b'_1, b'_2\}$, $b'_1 \in B[b_1, b_4]$, $b'_2 \in B[b_7, b_2]$, *and* G_0 *has a path from* a_0 *to* $B(b'_1, b'_2)$ *through* a'_0 *and internally disjoint from* B.

(ii) $V(G_1) \cap V(G_2) = \{b'_1, b'_2\}$, $b'_1 \in B[b_1, b_4]$, *and* $b'_2 \in B[b_7, b_2]$.

(iii) $V(G_1) \cap V(G_2) = \{b'_1, b'_2\}$, $b'_1 \in B[b_1, b_4]$, $b'_2 \in B[b_6, b_7)$, *and* G *has no edge from* $B(b'_2, b_7)$ *to* $A - a_7$.

(iv) $V(G_1) \cap V(G_2) = \{b'_1, b'_2\}$, $b'_1 \in B(b_4, b_5]$, $b'_2 \in B[b_7, b_2]$, *and* G *has no edge from* $B(b_4, b'_1)$ *to* $A - a_4$.

Proof. By Lemma 6.0.3, G_0 has a separation (G_1, G_2) such that $V(G_1) \cap V(G_2) = \{b'_1, b'_2\}$, $B[b'_1, b'_2] \subseteq G_1$, $\{a_0, b_1, b_2\} \subseteq G_2$, and one of the following holds:

(A) $b'_1 \in B[b_1, b_4]$, $b'_2 \in B[b_6, b_7)$, and there exists $e_8 = a_8 b_8 \in E(G)$ with $a_8 \in V(A - a_7)$ and $b_8 \in V(B(b'_2, b_7))$, or

(B) $b'_1 \in B(b_4, b_5]$, $b'_2 \in B[b_7, b_2]$, and there exists $e_8 = a_8 b_8 \in E(G)$ with $a_8 \in V(A - a_4)$ and $b_8 \in V(B(b_4, b'_1))$.

So we consider two cases.

Case 1. (A) holds.

We choose $\{b'_1, b'_2\}$ so that $B[b'_1, b_4]$ is minimal and, subject to this, $B[b'_2, b_7]$ is minimal. We also choose e_8 so that $A[a_8, a_5]$ is minimal. Note that $a_8 \in A[a_5, a_7)$, for otherwise, $(e_3, e_4, e_5, e_8, e_7)$ contradicts $\mathcal{P}$. So $a_5 \neq a_7$.

We consider two subcases according to the positions of a_5 and a_7.

Subcase 1.1. $a_5 \in A(a_7, a_2]$.

First, we note that for $e = ab \in E(G)$ with $a \in V(A)$ and $b \in V(B)$, if $a \in A(a_5, a_2]$ and $b \in B(b_1, b'_1)$, then (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$, and if $a \in A(a_8, a_5]$ or $b \in B[b_1, b_3)$ then e, e_3, e_8, e_4 form a doublecross. So we have

(1.1.1) G has no edge from $A(a_5, a_2]$ to $B(b_1, b_1')$, and G has no edge from $A(a_8, a_5]$ to $B[b_1, b_3)$.

(1.1.2) G has no edge from $A(a_8, a_2]$ to $B(b_2', b_2) + b_1$.

For, let $e = ab$ with $a \in A(a_8, a_2]$ and $b \in B(b_2', b_2) + b_1$.

Suppose $b = b_1$. Then $a \neq a_2$, and let $e_2 = a_2b' \in E(G)$ with $b' \in B(b_1, b_2)$. Then $b' \in B[b_7, b_2)$ to avoid the doublecross e, e_3, e_7, e_2. But now (e_2, e_7, e_5, e_3, e) contradicts the choice of $\mathcal{P}$.

So $b \in B(b_2', b_2)$. If $b \in B(b_2', b_7)$ then $a \in A(a_5, a_2]$ (by the minimality of $A[a_8, a_5]$); but then (e_3, e_4, e_5, e, e_7) contradicts the choice of $\mathcal{P}$. So $b \in B[b_7, b_2)$. Then $a \in A(a_5, a_2]$, as otherwise (e_3, e_4, e_5, e_6, e) contradicts the choice of $\mathcal{P}$. Now (e, e_7, e_8, e_6, e_5) is a 5-edge configuration. By Lemma 2.0.9 and 6.0.3, G_0 has a separation, which admits (i) or (ii), or contradicts the choice of $\{b_1', b_2'\}$. $\square$

(1.1.3) G has no edge from $A(a_7, a_2]$ to b_2.

To prove (1.1.3), let $e = ab_2 \in E(G)$ with $a \in A(a_7, a_2]$. We claim that $a \in A(a_5, a_4)$. To see this, first note that $a \neq a_2$. Moreover, $a \in A(a_5, a_2)$; as otherwise, (e_3, e_4, e_5, e_6, e) contradicts the choice of $\mathcal{P}$. Now suppose to the contrary that $a \notin A(a_5, a_4)$. Then $a \in A[a_4, a_2)$, and let $e_2 = a_2b_2' \in E(G)$ with $b_2' \in V(B) - \{b_1, b_2\}$. So $b_2' \in B(b_1, b_4]$ to avoid the doublecross e_2, e, e_4, e_8. But then $(e_3, e_2, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

Thus $b_7 = b_2$, or else (e_3, e_4, e_5, e_7, e) contradicts the choice of $\mathcal{P}$. Moreover, $a_8 = a_5$, or else (e_3, e_4, e_5, e_8, e) contradicts the choice of $\mathcal{P}$.

Suppose $a_6 \in A[a_1, a_7)$. Let $e_7' = a_7b_7' \in E(G)$ with $b_7' \in V(B - b_7)$. Then $b_7' \notin B[b_1, b_6)$ to avoid the doublecross e_6, e_7', e_7, e_8. Also $b_7' \neq b_6$ as otherwise $(e_3, e_4, e_7', e_8, e_7)$ contradicts the choice of $\mathcal{P}$. So $b_7' \in B(b_6, b_2)$. Then $(e_3, e_4, e_5, e_7', e_7)$ contradicts the choice of $\mathcal{P}$.

So $a_6 \in A(a_5, a_2]$ for all choices of e_6. Then $a_6 \in A[a_4, a_2]$, or else (e_3, e_4, e_6, e_8, e) contradicts the choice of $\mathcal{P}$. Let $e' = ab' \in E(G)$ with $b' \in V(B - b_2)$. Then $b' \neq b_6$ as

$a_6 \in A[a_4, a_2]$ for all choices of e_6. So $b' \in B(b_6, b_2)$ to avoid the doublecross e_8, e_6, e, e'. But then (e_3, e_4, e_5, e', e_7) contradicts the choice of $\mathcal{P}$. □

(1.1.4) There exists $e_9 = a_9b_9 \in E(G)$ with $a_9 \in A[a_1, a_8)$ and $b_9 \in B(b'_1, b'_2]$.

For, suppose such an edge does not exist. Then $a_6 \in A(a_5, a_2]$ and G has no edge from $B(b_4, b_5]$ to $A[a_1, a_5)$ by the choice of $\mathcal{P}$. Note that we have $a_5 \neq a_7$ and $a_7 \in A[a_1, a_5]$ and that, by (1.1.2) and (1.1.3), G has no edge from $B[b_7, b_2]$ to $A(a_5, a_2]$. Thus, we may apply Lemma 6.0.4, a contradiction. □

(1.1.5) $b_9 \in B(b_4, b'_2]$, $a_9 = a_3$, and so all edges from $B(b'_1, b'_2]$ to $A[a_1, a_8)$ must be from $B(b_4, b'_2]$ to a_3.

First, suppose $b_9 \in B(b'_1, b_4]$. Then $(e_9, e_4, e_5, e_6, e_8)$ is 5-edge configuration. Thus, by Lemma 2.0.9 and 6.0.3, G_0 has a separation, which admits (i) or (ii), or contradicts the choice of $\{b'_1, b'_2\}$. So we may assume $b_9 \in B(b_4, b'_2]$. Suppose $a_9 \neq a_3$. Then $a_9 \in A(a_3, a_4)$, to avoid the doublecross e_3, e_9, e_5, e_7. But now $(e_3, e_4, e_9, e_8, e_7)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. □

(1.1.6) $a_4 = a_2$.

Suppose $a_4 \neq a_2$. Let $e_2^* = a_2b_2^* \in E(G)$ with $b_2^* \in V(B)$. Then $b_2^* \in B(b_1, b_4]$ to avoid the doublecross e_2^*, e_4, e_9, e_8. Now (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. □

Thus, G has no edge e from $B[b_1, b'_1)$ to $v \in V(A(a_8, a_2])$; for, if $v \neq a_2$ then e, e_9, e_8, e_4 would form a doublecross, and if $v = a_2$ then (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. Hence, by (1.1.2) and (1.1.5), G has a 5-separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{b'_1, b'_2, a_8, a_3, a_2\}$, $V(A[a_8, a_2]) \cup V(B[b'_1, b'_2]) \cup \{a_3\} \subseteq V(H_1)$, and $V(A[a_3, a_8]) \cup \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_2)$, a contradiction.

Subcase 1.2. $a_5 \in A[a_1, a_7)$.

Then $a_6 \notin A(a_4, a_2)$ to avoid the doublecross e_4, e_6, e_5, e_7, and $a_6 \notin A(a_7, a_4)$ as, otherwise, $(e_3, e_4, e_6, e_8, e_7)$ contradicts the choice of $\mathcal{P}$. Hence, (1.2.1) $a_6 \in A[a_1, a_5)$ or $a_6 = a_4$.

(1.2.2) There exists $v \in \{a_4, b_4\}$ such that all edges from $A(a_8, a_2]$ to $B(b'_1, b'_2]$ are incident with v.

To prove (1.2.2), we first claim that G has no edge from $A(a_8, a_2] - a_4$ to $B(b'_1, b'_2] - b_4$. For otherwise, suppose there exists $e_9 = a_9b_9 \in E(G)$ with $a_9 \in A(a_8, a_2] - a_4$ to $b_9 \in B(b'_1, b'_2] - b_4$. If $b_9 \in B(b'_1, b_4)$ then $a_9 \in A(a_4, a_2]$ to avoid the doublecorss e_9, e_4, e_7, e_8; so $(e_3, e_9, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. Hence, $b_9 \in B(b_4, b'_2)$. Then $a_9 \in A(a_8, a_4)$ to avoid the doublecross e_4, e_9, e_8, e_7. Now $(e_3, e_4, e_9, e_8, e_7)$ contradicts the choice of $\mathcal{P}$.

Next, we see that either no edge is from b_4 to $A(a_8, a_2] - a_4$, or no edge is from a_4 to $B(b'_1, b'_2] - b_4$. In fact, by the choice of $\mathcal{P}$, any edge from b_4 to $A(a_8, a_2] - a_4$ must end in $A(a_8, a_4)$, and any edge from a_4 to $B(b'_1, b'_2] - b_4$ must end in $B(b_4, b'_2]$. If G has an edge from b_4 to $A(a_8, a_2] - a_4$ and an edge from a_4 to $B(b'_1, b'_2] - b_4$, then such two edges and e_7, e_8 form a doublecross in γ, a contradiction. □

Define $a'_1 \in A[a_1, a_8]$ such that G has no edge from $A[a_1, a'_1)$ to $B(b'_1, b'_2]$ and, subject to this, $A[a_1, a'_1]$ is maximal. By the definition of a'_1, G has an edge e_1 from a'_1 to $b \in B(b'_1, b'_2]$.

We claim that $a'_1 \in A[a_3, a_8)$. For otherwise, $a'_1 \in A[a_1, a_3)$. Now, if $b \in B(b_3, b'_2]$ then e_1, e_3, e_4, e_8 form a doublecross; if $b \in B(b_1, b_3]$ then $(e_1, e_4, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

So $a_1, a_3, a'_1, a_5, a_8, a_7, a_4, a_2$ occur on A in order.

(1.2.3) G has no edge from $A(a'_1, a_8)$ to $B - B[b'_1, b'_2]$.

For, otherwise, $a'_1 \neq a_8$, and there exists $e_9 = a_9b_9 \in E(G)$ with $a_9 \in A(a'_1, a_8)$ to $b_9 \in B - B[b'_1, b'_2]$. Then $b_9 \notin B[b_1, b'_1)$ to avoid the doublecross e_1, e_9, e_4, e_7.

We claim $b_9 = b_2$ and $a_9 \notin A[a_5, a_8)$. For, if $b_9 \in B(b'_2, b_7)$ then $a_9 \in A(a'_1, a_5)$ by the choice of e_8 (that $A[a_5, a_8]$ is minimal); now $(e_3, e_4, e_5, e_9, e_7)$ contradicts the choice of $\mathcal{P}$. Hence, $b_9 \in B[b_7, b_2]$. Thus, $a_9 \notin A[a_5, a_8)$; as otherwise $(e_3, e_4, e_5, e_8, e_9)$ contradicts the choice of $\mathcal{P}$. Now suppose $b_9 \neq b_2$. Then $(e_7, e_9, e_8, e_6, e_5)$ is a 5-edge configuration. Thus, by Lemma 2.0.9 and 6.0.3, G_0 has a separation, which admits (i) or (ii), or contradicts the choice of $\{b'_1, b'_2\}$.

We see $a_8 = a_5$; otherwise, $(e_3, e_4, e_5, e_8, e_9)$ contradicts the choice of $\mathcal{P}$. Moreover, $a_4 = a_2$; for otherwise, G has an edge e' from a_2 to B, then either (e_3, e', e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$ or e', e_4, e_5, e_7 form a doublecross.

Next, we claim that all edges from $A(a_8, a_2)$ to B must end in $\{b_4, b_2\}$. First, G has no edge from $A(a_8, a_2)$ to b_1; otherwise, such an edge together with e_7, e_3, e_4 forms a doublecross. G has no edge from $A(a_8, a_2)$ to $B(b_1, b_4)$; otherwise, such an edge together with e_3, e_5, e_1, e_9 forms a 5-edge configuration contradicting the choice of $\mathcal{P}$. G has no edge from $A(a_8, a_2)$ to $B(b_4, b_8)$; otherwise, such an edge together with e_3, e_4, e_8, e_7 forms a 5-edge configuration contradicting the choice of $\mathcal{P}$. G has no edge from $A(a_8, a_2)$ to $B[b_8, b_2)$; otherwise, such an edge together with e_3, e_4, e_5, e_9 forms a 5-edge configuration contradicting the choice of $\mathcal{P}$.

Therefore, since $a_7 \in A(a_8, a_2)$, then $\{a_2, a_8, b_2, b_4\}$ is a 4-cut in G separating a_7 from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. $\square$

By (1.2.2) and (1.2.3), G has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{b'_1, b'_2, a_8, a'_1, v\}$, $b_5 \in V(H_2 - H_1)$, and $\{a_0, a_1, a_2, b_1, b_2\} \subseteq H_1$, a contradiction.

Case 2. (B) holds.

We choose (G_1, G_2) such that $B[b'_1, b'_2]$ is maximal.

We claim that $a_8 \in A[a_1, a_3] \cup A(a_4, a_2]$. For, suppose $a_8 \in A(a_3, a_4)$. Then $a_6 \in A[a_7, a_8]$ and $a_8 \notin A[a_7, a_5]$; for otherwise $(e_3, e_4, e_8, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. Therefore, $a_5 \notin A[a_6, a_8]$ (since $a_6 \notin A[a_5, a_7]$). So $(e_3, e_4, e_8, e_5, e_6)$ is a 5-edge configuration. Thus, by Lemma 2.0.9 and 6.0.3, G_0 has a separation, which admits (i) or (ii), or contradicts the choice of $\{b'_1, b'_2\}$.

Therefore, we have distinguished two cases.

Subcase 2.1. $a_8 \in A(a_4, a_2]$.

Choose e_8 so that $A[a_8, a_2]$ is minimal. Note that $a_6 \in A[a_8, a_2]$ and $a_7 \in A(a_3, a_5]$, since, otherwise, e_4, e_8 and two of $\{e_5, e_6, e_7\}$ force a doublecross.

(2.1.1) G has no edge from $A(a_5, a_2]$ to $B[b_1, b_4) \cup B(b_6, b_2]$.

For, let $e = ab \in E(G)$ with $a \in A(a_5, a_2]$ and $b \in B[b_1, b_4) \cup B(b_6, b_2]$.

Suppose $b \in B(b_6, b_2]$. Then $a \in A[a_8, a_2]$ to avoid the doublecross e, e_4, e_5, e_8. So $b \in B[b_7, b_2]$, or else (e_3, e_4, e_5, e, e_7) contradicts the choice of $\mathcal{P}$. Suppose $b = b_2$. Then $a \neq a_2$, so there exists $e' = a_2b' \in E(G)$ with $b' \in B(b_1, b_2)$. Now $b' \in B(b_1, b_4]$ to avoid the doublecross e_4, e_5, e, e'. But then, (e_3, e', e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. Thus, $b \neq b_2$. Now (e, e_7, e_5, e_8, e_4) is a 5-edge configuration. Hence, by Lemma 2.0.9 and 6.0.3, G_0 has a separation, which admits (i) or (ii), or contradicts the choice of $\{b'_1, b'_2\}$.

Thus, $b \in B[b_1, b_4)$ for every choice of $e = ab$. Assume $a \in A[a_4, a_2]$. Then $b = b_1$, or else, (e, e_3, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. Now, since G has no edge from $B(b_6, b_2]$ to $A(a_5, a_2]$, G has an edge from a_2 to $B(b_1, b_7)$, which together with e, e_3, e_7 forms a doublecross. So $a \in A(a_5, a_4)$. Then either e_3, e_4, e_5, e form a doublecross, or (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. $\square$

(2.1.2) G has no edge from $B(b_1, b_3)$ to A.

For otherwise, let $e = ab \in E(G)$ with $a \in A$ and $b \in B(b_1, b_3)$. If $a \in A[a_1, a_3]$, then (e, e_4, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$; if $a \in A(a_3, a_4)$, then e, e_3, e_4, e_7 form a doublecross; if $a \in A[a_4, a_2]$, then (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. Hence, (2.1.2) holds.$\square$

(2.1.3) $b'_2 = b_2$ and G_0 has a separation (G'_1, G'_2) that $V(G'_1 \cap G'_2) = \{b_1, b''_2, a_0\}$, $b''_2 \in B(b'_1, b'_2)$, $B[b_1, b''_2] \subseteq G'_1$, and $\{a_0, b_1, b_2\} \subseteq V(G'_2)$.

For, otherwise, we claim that there exists $v \in \{a_5, b_5\}$ such that all edges from $B(b'_1, b'_2)$ to $A[a_1, a_8)$ in G contain v. To prove this, we first assume that $b_5 \in B(b'_1, b'_2)$, and there exist $e_5' = a_5 b_5', e_5'' = a_5' b_5 \in E(G)$ with $a_5' \in A[a_1, a_8)$ and $b_5' \in B(b_1', b_2')$ such that $a_5' \neq a_5$ and $b_5' \neq b_5$. Then e_5', e_5'' form a cross to avoid the doublecross e_5', e_5'', e_4, e_8. But now, $b_5' \in B(b_5, b_2')$ by the choice of $\mathcal{P}$, and so $(e_6, e_5', e_5'', e_8, e_4)$ is a 5-edge configuration; and since (i) or (ii) does not hold, then (2.1.3) follows from Lemmas 2.0.9 and 6.0.3 and the choice of $\{b_1' b_2'\}$. So assume that such e_5', e_5'' do not exist. Therefore, if the claimed v does not exist, then there exists $e = ab \in E(G)$ such that $a \in A[a_1, a_8) - a_5$ and $b \in B(b_1', b_2') - b_5$. If $b \in B(b_1', b_5)$ then $a \in A(a_5, a_8)$ to avoid the doublcross e, e_5, e_4, e_8. Hence, (e_6, e_5, e, e_8, e_4) is a 5-edge configuration and (2.1.3) follows from Lemmas 2.0.9 and 6.0.3 and the choice of $\{b_1' b_2'\}$; or else, (i) or (ii) holds. So $b \in B(b_5, b_2')$. Then $a \notin A(a_5, a_8)$ to avoid the doublecross e_4, e_5, e_8, e. Hence, (e, e_6, e_5, e_8, e_4) is a 5-edge configuration and (2.1.3) follows from Lemmas 2.0.9 and 6.0.3 and the choice of $\{b_1' b_2'\}$; or else, (i) or (ii) holds.

Now, we see that G has no edge from $A(a_8, a_2)$ to $B - B[b_1', b_2']$. For otherwise, by our claim, G has an edge $e = ab$ with $a \in A(a_8, a_2)$ and $b \in B - [b_1', b_2']$. By (2.1.1), $b \in B[b_4, b_1')$. By the choice of $\mathcal{P}$, $b \neq b_4$. So $b \in B(b_4, b_1')$, which contradicts the choice of e_8.

Thus, G has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{v, a_8, a_2, b_1', b_2'\}$, $a_0, a_1, a_2, b_1, b_2 \in V(H_1)$, and $b_6 \in V(H_2 - H_1)$, a contradiction. □

By (2.1.3), $\alpha(A, B) \leq 1$. We may assume

(2.1.4) $b_2'' \notin B[b_7, b_2]$, and either $b_7 = b_2$ (in which case let $B_0 = B[b_2'', b_2]$) or $b_7 \neq b_2$ and $G_0 - (B[b_1, b_2'') \cup B[b_7, b_2))$ has a path B_0 from b_2'' to b_2.

Clearly, $b_2'' \notin B[b_7, b_2]$ as otherwise the conclusion of the lemma holds. Now suppose $b_7 \neq b_2$ and the desired path B_0 in $G_0 - (B[b_1, b_2'') \cup B[b_7, b_2))$ does not exist. Then there exist $b_2^* \in V(B[b_7, b_2))$ and a separation (H_1, H_2) in G_0 such that $V(H_1 \cap H_2) = \{b_1, b_2^*, a_0\}$. This implies the conclusion of this lemma, a contradiction. □

We choose b_2'' so that $B[b_2'', b_7]$ is minimal.

(2.1.5) G has two non-incident edges from $B(b_1', b_2]$ to $A[a_1, a_5]$.

For otherwise, $b_1' = b_5$, and there exists $v \in \{a_7, b_7\}$ such that all edges in G from $B(b_1', b_2]$ to $A[a_1, a_5]$ are incident with v.

G has no edge from $B(b_1', b_6]$ to $A(a_5, a_8)$. For otherwise, let $e = ab \in E(G)$ with $b \in B(b_1', b_6]$ and $a \in A(a_5, a_8)$. Now, since $b_1' = b_5$, then e, e_4, e_5, e_8 form a doublecross, a contradiction.

G has no edge from $A(a_8, a_2]$ to $B[b_4, b_1')$. For otherwise, let $e = ab \in E(G)$ with $b \in B[b_4, b_1')$ and $a \in A(a_8, a_2]$. By the choice of e_8, $b \notin B(b_4, b_1')$, and so $b = b_4$. But then, (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$.

Now, combined with (2.1.1), G has a separation (H_1, H_2) of order 5, such that $V(H_1 \cap H_2) = \{v, b_1', b_2, a_8, a_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_1)$, and $V(B'[b_1', b_2] \cup A[a_8, a_2]) \subseteq V(H_2)$, a contradiction. $\square$

Note that no two edges of G from $B(b_1', b_2]$ to $A[a_1, a_4]$ can be parallel, as such edges would form a doublecross with e_4, e_8. Therefore, there exist two non-incident edges $e_9' = a_9'b_9'$, $e_9'' = a_9''b_9''$ with $a_9', a_9'' \in A[a_1, a_5]$ and $b_9', b_9'' \in B(b_1', b_2]$ such that b_1, b_9', b_9'' occur on B in order, and a_1, a_9'', a_9' occur on A in order. Moreover, we further choose e_9', e_9'' so that $A[a_9', a_2] \cup B[b_9'', b_2]$ is minimal. By the existence of e_7, we have $a_9' \in A[a_7, a_2]$ and $b_9'' \in B[b_7, b_2]$.

(2.1.6) G has two parallel edges e', e'' from $b', b'' \in B(b_3, b_1')$ to $a', a'' \in A[a_4, a_2]$, with b_1, b', b'', b_2 occurring on B in order.

Suppose it fails. Then $b_3 = b_4$ as otherwise e_4, e_8 give the desired edges for (2.1.6). Let $e = a_1b \in E(G)$ with $b \notin \{b_1, b_2, b_3, b_7\}$. Then $b \notin B(b_1, b_3)$; otherwise, (e, e_4, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. Moreover, $b \notin B(b_3, b_7)$ to avoid the doublecross e, e_4, e_7, e_8. So $b \in B(b_7, b_2)$.

Now, since (e, e_6, e_5, e_8, e_4) is a 5-edge configuration, then $b_2'' \in B[b_6, b_7)$; or else, by Lemma 2.0.9 and 6.0.3, (i) or (ii) holds, or it contradicts the choice of b_1', b_2' or contradicts the choice of b_2''.

Now, let $a^* \in A[a_1, a_2]$, such that G has an edge e^* from $b^* \in B(b_2'', b_7) \cup B(b_7, b_2)$ to a^*, subject to this, $A[a^*, a_2]$ is minimal, and subject to this, $B[b_2'', b^*]$ is minimal.

We claim that $a^* \notin A(a_5, a_2]$. For otherwise, suppose $a^* \in A(a_5, a_2]$. Now, if $b^* \in B(b_2'', b_7)$, then $(e_3, e_4, e_5, e^*, e_7)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. So $b^* \in B(b_7, b_2)$. If $a^* \in A(a_5, a_8)$, then e_4, e_5, e_8, e^* form a doublecross; if $a^* \in A[a_8, a_2]$, then (e, e^*, e_5, e_8, e_4) is a 5-edge configuration contradicting the choice of $\mathcal{P}$. This finishes our claim.

Now, we further claim that G has no edge from $A(a_1, a^*)$ to $B[b_1, b_3) \cup B(b_3, b_2'')$. For otherwise, let $e' = a'b' \in E(G)$ with $a' \in A(a_1, a^*)$ and $b' \in B[b_1, b_3) \cup B(b_3, b_2'')$. Then $b' \notin B(b_3, b_2'')$ to avoid the doublecross e_4, e_8, e', e^*. So $b' \in B[b_1, b_3)$. But then $a' \notin A(a_3, a^*)$ to avoid the doublecross e_3, e_4, e', e_7. So $a' \in A[a_1, a_3]$, and (e', e_4, e_5, e_6, e_7) is a 5-edge configuration contradicting the choice of $\mathcal{P}$.

We may assume G has an edge e_7' from b_7 to $a_7' \in A(a^*, a_2]$ and an edge e_3' from b_3 to $a_3' \in A(a_1, a^*)$. For otherwise, G has a separation (H_1, H_2) of order 5, such that $V(H_1 \cap H_2) = \{a_1, a^*, v, b_2'', b_2\}$, $v \in \{b_3, b_7\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_1)$, and $V(A[a_1, a^*] \cup B[b_2'', b_2]) \subseteq V(H_2)$, a contradiction.

Now, G has a separation (H_1, H_2) of order 6, such that $V(H_1 \cap H_2) = \{a_1, a^*, b_3, b_2'', b_7, b_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_1)$, and $V(A[a_1, a^*] \cup B[b_2'', b_2]) \subseteq V(H_2)$.

We see that any two edges from $A[a_1, a^*]$ to $B[b_2'', b_2]$ are not parallel; or else, such two edges together with e_4, e_8 form a doublecross. Moreover, by the choice of $\mathcal{P}$, we can further assume $a_7' \in A(a_5, a_2]$.

Now, assume $b^* \notin B(b_2'', b_7)$. Then since any two edges from $A[a_1, a^*]$ to $B[b_2'', b_2]$ are not parallel, then, combined with the choice of e^*, we have $(H_2, a_1, b_3, a^*, b_7, b_2'', b_2)$ is planar, a contradiction to Lemma 2.0.3.

So $b^* \in B(b_2'', b_7)$. But then $(e_7', e, e^*, e_6, e_3')$ is a 5-edge configuration. Now, by Lemma 2.0.9 and 6.0.3, (i) or (ii) holds, or it contradicts the choice of b_2''. □

We choose e', e'' in (2.1.5) such that $B[b_3, b']$ is minimal and, subject to this, $B[b'', b'_1]$ is minimal.

Suppose $G_0 - B(b_1, b_3] - B(b'_1, b_2]$ has disjoint paths P_1, P_2 from b_1, a_0 to b', b'', respectively. Let $A' := P_2 \cup e'' \cup A[a'', a_2]$ and $B' := P_1 \cup e' \cup A[a'_9, a'] \cup e'_9 \cup B[b'_9, b''_2] \cup B_0$. Now, since A, B is a good frame, then the existence of A', B', $A[a_1, a''_9] \cup e''_9 \cup B[b''_9, b_2]$, and $A[a_1, a_3] \cup e_3 \cup B[b_1, b_3]$ shows $\alpha(A, B) = 2$, a contradiction.

Thus, such P_1, P_2 do not exist. Then G_0 has a separation (H_1, H_2) with $V(H_1 \cap H_2) = \{b^*_1, b^*_2\}$ such that $b^*_1 \in B(b_1, b_3]$, $B[b^*_1, b''] \subseteq H_1$, and $\{a_0, b_1, b_2\} \subseteq H_2$. We may assume $b^*_2 \in B[b'', b'_1)$ as otherwise we have (i).

Now, suppose G has no edge from $B(b^*_2, b'_1)$ to A, then, combined with (2.1.2), G has a separation (K_1, K_2) of order 5, such that $V(K_1 \cap K_2) = \{b_1, b^*_1, b^*_2, b'_1, a_0\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(K_1)$, and $V(B[b_1, b^*_1] \cup B[b^*_2, b'_1]) \subseteq V(K_2)$, a contradiction.

So we may assume G has an edge e_0 from $b_0 \in B(b^*_2, b'_1)$ to $a_0 \in A$. By the choice of e', e'', $a_0 \in A[a_4, a'')$. So $(e_3, e'', e_0, e_6, e_7)$ is a 5-edge configuration. Now, by Lemma 2.0.9 and 6.0.3, and by the existence of $\{b^*_1, b^*_2\}$, (i) or (ii) holds, or it contradicts the choice of b'_1, b'_2.

Subcase 2.2 $a_8 \in A[a_1, a_3]$.

Note that if $b_3 = b_4$ we have symmetry between e_3 and e_4, so by Subcase 2.1, we may assume that

(2.2.1) if $b_3 = b_4$ then there exists $e_9 = a_4 b_9 \in E(G)$ with $b_9 \in B(b_4, b'_1)$.

(2.2.2) The following holds: (a) G has no edge from $B(b_4, b_7)$ to $A(a_4, a_2]$ and so $a_6 \notin A(a_4, a_2]$; and (b) G has no edge from $B(b_3, b_7)$ to $A[a_1, a_3)$ and so $a_8 = a_3$.

We have (a) to avoid a doublecross formed by such an edge and e_4, e_7, e_8. Now suppose (b) fails, and let e' be an edge from $B(b_3, b_7)$ to $A[a_1, a_3)$. If $b_3 \neq b_4$, e_3, e_4, e', e_7 form a doublecross. So $b_3 = b_4$. Then by (2.2.1), e_3, e_9, e', e_7 form a doublecross.

(2.2.3) There exists $v \in \{a_4, b_4\}$ such that all edges from $B[b_1, b_1')$ to $A(a_3, a_2]$ must contain v.

Since we have proved $a_8 \in A[a_1, a_3] \cup A(a_4, a_2]$, then, combined with (2.2.2), all edges from $B(b_4, b_1')$ to A must end in $\{a_3, a_4\}$.

Now, we claim that G has no edge from $B[b_1, b_4)$ to $A(a_3, a_2]$. For, let $e = ab \in E(G)$ with $b \in B[b_1, b_4)$ and $a \in A(a_3, a_2]$. Then $a \in A[a_4, a_2]$, to avoid the doublecross e, e_4, e_5, e_8. So $b = b_1$ by the choice of $\mathcal{P}$. Then $a \neq a_2$; so G has an edge $e_2 = a_2 b'$ with $b' \in B(b_1, b_2)$. Then $b' \in B[b_7, b_2)$ to avoid the doublecross e_2, e_7, e_8, e'. If $b_3 \neq b_4$ then (e_2, e_7, e_4, e_3, e) contradicts the choice of $\mathcal{P}$. So $b_3 = b_4$. Then e_9 is defined by (2.2.1). Hence, (e_2, e_7, e_9, e_3, e) contradicts the choice of $\mathcal{P}$.

Thus, if (2.2.3) fails, then there exist $e' = a_4 b', e'' = a'' b_4$ with $a'' \in A(a_3, a_2] - a_4$ and $b' \in B(b_4, b_1')$. By the choice of $\mathcal{P}$, $a'' \in A(a_3, a_4)$. So e_8, e', e'', e_7 form a doublecross, a contradiction.

(2.2.4) $a_1 = a_3$.

For, suppose $a_1 \neq a_3$. Then there exists $e_1 = a_1 b \in E(G)$ with $b \in B(b_1, b_2)$. Indeed, $b \notin B(b_1, b_4]$ by the choice of $\mathcal{P}$; $b \notin B(b_4, b_7)$ by (2.2.2). So $b \in B[b_7, b_2)$. Moreover, $b_3 = b_4$, for, otherwise, $(e_1, e_7, e_8, e_4, e_3)$ contradicts the choice of $\mathcal{P}$. Thus e_9 in (2.2.1) is defined.

We claim that G has no edge from $B[b_1', b_7)$ to $A(a_1, a_7)$. For such an edge and e_1, e_7, e_9, e_3 form a 5-edge configuration. Hence, by Lemma 2.0.9 and 6.0.3, (i) or (ii) holds, or it contradicts the choice of b_1', b_2'.

Thus, combined with (2.2.2), $a_6 \in A(a_7, a_4]$. So $(e_6, e_1, e_5, e_9, e_3)$ is a 5-edge configuration. Hence, $b_2' = b_2$; or else, by Lemma 2.0.9 and 6.0.3, (i) or (ii) holds, or it contradicts the choice of b_1', b_2'.

Now, we claim that $\{b_1, b_2, b_1', a_3, a_4\}$ is a cut in G separating $\{a_1, a_2\}$ from $\{a_0\}$, a contradiction to that G^* is 6-connected. In fact, since $b_2' = b_2$, we just need to show that G

has no edge from $B(b_1, b_1')$ to $A[a_1, a_2] - \{a_3, a_4\}$. Suppose there exists $e^* = a^*b^* \in E(G)$ with $a^* \in A[a_1, a_2] - \{a_3, a_4\}$ and $b^* \in B(b_1, b_1')$. By (2.2.3) and by the existence of e_9, $a^* \in A[a_1, a_3)$. Then $b^* \notin B(b_1, b_3]$; otherwise, $(e^*, e_4, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. But then, $b^* \in B(b_3, b_1')$, and e^*, e_3, e_6, e_7 form a doublecross. Hence, our claim is true, which finishes the proof of (2.2.4). □

Let $e_2 = a_2'b' \in E(G)$ with $a_2' \in V(A)$ and $b' \in B(b_1', b_2')$, such that $A[a_2, a_2']$ is minimal. We may assume that

(2.2.5) there exists $e_0 = a_0b_0 \in E(G)$ with $a_0 \in A(a_1, a_2')$ and $b_0 \in B - B[b_1', b_2']$.

For otherwise, G has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{b_1', b_2', a_1, a_2'\}$ with $a_0, a_1, a_2, b_1, b_2 \in V(H_1)$, and $b_6 \in V(H_2)$, a contradiction. □

(2.2.6) $b_0 \in B[b_1, b_1')$ for every choice of e_0.

For, otherwise, $b_0 \in B(b_2', b_2]$. Then $a_0 \in A(a_1, a_4)$, to avoid the doublecross e_4, e_8, e_2, e_0. Also, $a_6 \in A[a_5, a_0]$; otherwise $(e_3, e_4, e_5, e_6, e_0)$ contradicts the choice of $\mathcal{P}$. Moreover, $a_7 \in A[a_6, a_0]$; or else $(e_3, e_4, e_6, e_7, e_0)$ contradicts the choice of $\mathcal{P}$. But this shows that $a_6 \in A[a_5, a_7]$, a contradiction. □

Now, combined with (2.2.3), G has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{a_1, a_2', b_1', b_2', v\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_1)$, and $V(A[a_1, a_2'] \cup B[b_1', b_2']) \subseteq V(H_2)$, a contradiction. □

Lemma 6.0.6 *Suppose γ is infeasible, and A', B' is a core a_0-frame in γ. Suppose $\mathcal{P} = (e_3, e_4, e_5, e_6, e_7)$ is an ideal 5-edge configuration w.r.t. an ideal frame A, B in γ. Then (i)–(iv) of Lemma 6.0.5 do not hold.*

Proof. For notation convenience, we assume a_1, a_3, a_4, a_2 occur on A in order, and $b_1, b_3, b_4, b_5, b_6, b_7, b_2$ occur on B in order. And we choose b_1', b_2' satisfying the conclusions of Lemma

6.0.5 so that b_1', b_2' are defined from (i) or (ii) whenever possible, subject to this, $B[b_1, b_1']$ is minimal, and subject to this, $B[b_7, b_2']$ is minimal.

Moreover, we define $t_0, t_1, t_2 \in V(G)$, $B(t_1, t_2)$, $B[b_1, t_1)$, and $B(t_2, b_2]$ according to $\{b_1', b_2'\}$.

(i) Suppose (i) of Lemma 6.0.5 occurs. We define $t_1 := b_1'$, $t_2 := b_2'$, $t_0 := a_0'$, $B(t_1, t_2) := B(b_1', b_2')$, $B[b_1, t_1) := B[b_1, b_1')$, and $B(t_2, b_2] := B(b_2', b_2]$.

(ii) Suppose (ii) of Lemma 6.0.5 occurs. We define $t_0 := t_1 := b_1'$, $t_2 := b_2'$, $t_0 := a_0'$, $B(t_1, t_2) := B(b_1', b_2')$, $B[b_1, t_1) := B[b_1, b_1')$, and $B(t_2, b_2] := B(b_2', b_2]$.

(iii) Suppose (iii) of Lemma 6.0.5 occurs.

(a) If G has no edge from $B(b_2', b_7)$ to A, we define $t_1 := b_1'$, $t_2 := b_7$, $t_0 := b_2'$, $B(t_1, t_2) := B(b_1', b_7)$, $B[b_1, t_1) := B[b_1, b_1')$, and $B(t_2, b_2] := B(b_7, b_2]$.

(b) If G has an edge f_7 from $b_7^* \in B(b_2', b_7)$ to a_7, we define $t_1 := b_1'$, $t_2 := a_7$, $t_0 := b_2'$, $B(t_1, t_2) := B(b_1', b_2']$, $B[b_1, t_1) := B[b_1, b_1')$, and $B(t_2, b_2] := B[b_7, b_2]$.

(iv) Suppose (iv) of Lemma 6.0.5 occurs.

(a) If G has no edge from $B(b_4, b_1')$ to A, we define $t_1 := b_4$, $t_2 := b_2'$, $t_0 := b_1'$, $B(t_1, t_2) := B(b_4, b_2')$, $B[b_1, t_1) := B[b_1, b_4)$, and $B(t_2, b_2] := B(b_2', b_2]$.

(b) If G has an edge f_4 from $b_4^* \in B(b_4, b_1')$ to a_4, we define $t_1 := a_4$, $t_2 := b_2'$, $t_0 := b_1'$, $B(t_1, t_2) := B[b_1', b_2')$, $B[b_1, t_1) := B[b_1, b_4]$, and $B(t_2, b_2] := B(b_2', b_2]$.

Note that by Lemma 6.0.5, when $(iii)(b)$ occurs, G has no edge from $B(b_2', b_7)$ to $A[a_1, a_2] - a_7$; when (iv)(b) occurs, G has no edge from $B(b_4, b_1')$ to $A[a_1, a_2] - a_4$.

Let a_1^*, a_2^* be vertices on A such that a_1, a_1^*, a_2^*, a_2 occur on A in that order, G has edges f_i, $i = 1, 2$, from a_i^* to $b_i^* \in B(t_1, t_2)$, and subject to these, $A[a_1^*, a_2^*]$ is maximal. Notice that $A[a_5, a_6] \subseteq A[a_1^*, a_2^*]$.

Case 1. G has no edge from $B[b_1, t_1)$ to $A(a_1^*, a_2^*)$, which is different from e_4.

Now, as G^* is 6-connected, $\{t_0, t_1, t_2, a_1^*, a_2^*\}$ is not a cut in G separating $V(A(a_1^*, a_2^*) \cup B(t_1, t_2))$ from $\{a_0, a_1, a_2, b_1, b_2\}$. Thus, combined with Lemma 6.0.5, G has an edge e_8 from $b_8 \in B(t_2, b_2]$ to $a_8 \in A(a_1^*, a_2^*)$ (or $a_8 \in A(a_1^*, a_2^*) - a_7$ if $(iii)(b)$ occurs). Then, obviously, $b_8 \in B(b_2', b_2] \cap B[b_7, b_2]$.

We first see that $a_8 \in A(a_3, a_4)$. For, suppose $a_8 \notin A(a_3, a_4)$. Hence, $a_8 \in A(a_1, a_3] \cup A[a_4, a_2)$. If $a_8 \in A(a_1, a_3]$, then $a_1^* \in A[a_1, a_3)$, and so e_3, f_1, e_5, e_8 force a doublecross, or $(f_1, e_4, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. Therefore, $a_8 \in A[a_4, a_2)$. Then $b_2^* \in B(b_1', b_4]$; otherwise e_4, e_5, f_2, e_8 force a doublecross. But now, $(e_3, f_2, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

We claim that $b_8 = b_7$, and so $(iii)(b)$ occurs with $a_8 \neq a_7$ and f_7 existing. For, suppose $b_8 \in B(b_7, b_2]$. Then $(e_3, e_4, e_5, e_6, e_8)$ (when $a_6 \notin A[a_5, a_8]$) or $(e_3, e_4, e_6, e_7, e_8)$ (when $a_6 \in A[a_5, a_8]$) contradicts the choice of $\mathcal{P}$.

Let $e = a_8 b \in E(G)$ with $b \in B[b_1, b_2] - \{b_4, b_7\}$. We also claim that $b \in B[b_1, b_4)$. First, by $b_8 = b_7$, $b \notin B(b_7, b_2]$. By (iii)(b), $b \notin B(b_2', b_7)$. Moreover, $b \notin B(b_4, b_2']$; otherwise, (e_3, e_4, e, f_7, e_8) is a 5-edge configuration contradicting the choice of $\mathcal{P}$. This finishes our claim.

Now, we may assume $a_8 \in A(a_3, a_7)$; otherwise, e, e_4, f_7, e_8 form a doublecross.

Then $a_7 \in A[a_1, a_5]$; otherwise, $(e_3, e_4, e_5, f_7, e_8)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$.

We see that $a_6 \in A(a_5, a_2]$. In fact, if $a_6 \in A[a_1, a_8)$, then e_4, e_6, e_8, e form a doublecross; if $a_6 \in A[a_8, a_7)$, then $(e_3, e_4, e_6, f_7, e_8)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$.

G has no edge from $B(b_4, b_5]$ to $A[a_1, a_5)$. For, otherwise, let $e_9 = a_9 b_9 \in E(G)$ with $a_9 \in A[a_1, a_5)$ and $b_9 \in B(b_4, b_5]$. Now, $a_9 \notin A[a_1, a_8)$ to avoid the doublecross e, e_4, e_8, e_9. Moreover, $a_9 \notin A[a_8, a_7)$ and $b_9 \notin B(b_4, b_5)$; or else, $(e_3, e_4, e_9, f_7, e_8)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. So $a_9 \in A[a_7, a_5)$ and $b_9 = b_5$, but then $(e_3, e_4, e_9, e_6, e_7)$ is a 5-edge configuration, contradicting the choice of $\mathcal{P}$.

G has no edge from $B[b_7, b_2]$ to $A(a_5, a_2]$. For, otherwise, let $e_9 = a_9b_9 \in E(G)$ with $a_9 \in A(a_5, a_2]$ and $b_9 \in B[b_7, b_2]$. Then $(e_9, e_8, f_7, e_6, e_5)$ forms a 5-edge configuration. Now, by Lemma 6.0.3, G_0 has a cut $\{b_1'', b_2''\}$ or $\{b_1'', b_2'', a_0''\}$ (w.r.t. $(e_9, e_8, f_7, e_6, e_5)$) satisfying the conclusion of Lemma 6.0.3, such that b_1, b_1'', b_2'', b_2 occur on B in order. But then, by Lemma 2.0.9, G_0 has a cut, which contradicts the choice of $\{b_1', b_2'\}$.

Now, the existence of $\{b_1', b_2'\}$ contradicts Lemma 6.0.4.

Case 2. G has an edge e_8 from $b_8 \in B[b_1, t_1)$ to $a_8 \in A(a_1^*, a_2^*)$, such that $e_8 \neq e_4$.

Note that $b_8 \in B[b_1, b_4] \cap B[b_1, b_1')$. Now, we distinguish two subcases, $a_8 \in A(a_5, a_2^*)$ and $a_8 \in A(a_1^*, a_5]$.

Subcase 2.1. $a_8 \in A(a_5, a_2^*)$.

We choose e_8 so that $A[a_8, a_2]$ is maximal.

(2.1.1) $b_8 \notin B(b_1, b_4)$.

For, otherwise, $b_8 \in B(b_1, b_4)$. Then $a_8 \notin A(a_5, a_7]$ to avoid the doublecross e_8, e_4, e_5, e_7. Now, $b_3 = b_4$ and $a_8 \in A[a_1, a_4)$; otherwise, $(e_3, e_8, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. But then, e_3, e_4, e_7, e_8 form a doublecross. □

(2.1.2) $b_8 \neq b_4$.

For, otherwise, $b_8 = b_4$, and $(iv)(b)$ occurs with f_4 existing.

We see that $a_8 \notin A(a_4, a_2]$; otherwise, $(e_3, e_8, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. So $a_8 \in A(a_5, a_4)$.

G has no edge from $A(a_5, a_4)$ to $B(b_5, b_2]$; otherwise, such an edge together with e_5, e_8, f_4 forms a doublecross. Hence, $a_7 \in A[a_1, a_5]$ and $a_6 \notin A(a_5, a_4)$. Moreover, $a_6 \notin A[a_1, a_7)$ to avoid the doublecross e_6, e_7, e_8, f_4. So $a_6 \in A[a_4, a_2]$.

Now, since a_8 has degree at least 6 in G, then G has an edge e_8' from a_8 to $b_8' \in B[b_1, b_2] - \{b_1, b_4, b_5\}$. Since G has no edge from $A(a_5, a_4)$ to $B(b_5, b_2]$, then, combined with (2.1.1), $b'_8 \in B(b_4, b_5)$. But then, $(e_3, e_4, e'_8, e_6, e_7)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. □

Hence, $b_8 = b_1$ and $b'_1 \neq b_1$. Now, $a_8 \in A[a_4, a_2^*)$ to avoid the doublecross e_8, e_4, e_3, e_7. And $b_2^* \in B[b_7, b'_2)$ to avoid the doublecross e_8, f_2, e_3, e_7. Moreover, $b_3 = b_4$; otherwise, $(f_2, e_7, e_4, e_3, e_8)$ contradicts the choice of $\mathcal{P}$.

Now, by no doublecross and by the choice of $\mathcal{P}$, G has no edge from $B(b_1, b_3)$ to A. Also, $a_5 \in A[a_1, a_7]$, or else $(f_2, e_7, e_5, e_3, e_8)$ contradicts the choice of $\mathcal{P}$.

Finally, $a_6 \in A[a_1, a_5)$, as, otherwise, e_8, e_6, e_3, e_7 (when $a_6 \in A(a_8, a_2]$) would form a doublecross , or $(f_2, e_7, e_6, e_3, e_8)$ (when $a_6 \in A[a_5, a_8]$) contradicts the choice of $\mathcal{P}$.

(2.1.3) G has no edge from $B(b_6, b_2]$ to $A[a_1, a_5)$ and no cross from $B[b_6, b_2]$ to $A[a_5, a_2]$.

For, let $e = ab \in E(G)$ with $b \in B(b_6, b_2]$ and $a \in A[a_1, a_5)$. Then $b = b_2$; or else, (e_3, e_4, e_5, e, e_7) (when $b \notin B(b_6, b_7)$) or (f_2, e, e_5, e_3, e_8) (when $b \in B[b_7, b_2)$) contradicts the choice of $\mathcal{P}$. But then $a \neq a_1$, and e, e_8 and two edges from a_1, a_2 to $B(b_1, b_2)$ would form a doublecross.

Moreover, suppose G has a cross from $B[b_6, b_2]$ to $A[a_5, a_2]$, then such a cross and e_5, e_6 would form a doublecross, a contradiction. □

(2.1.4) G has no edge from $B(b_1, b_3)$ to A.

For, otherwise, let $e = ab \in E(G)$ with $a \in A$ and $b \in B(b_1, b_3)$. Then $a \in A[a_4, a_8]$; or else, (f_2, e_7, e_4, e, e_8) contradicts the choice of $\mathcal{P}$. But now, (e, e_3, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. □

(2.1.5) G has no edge from $A(a_4, a_2]$ to $B(b_1, b_7)$.

For, otherwise, let $e = ab \in E(G)$ with $a \in A(a_4, a_2]$ and $b \in B(b_1, b_7)$. Then $b \notin B(b_4, b_7)$ to avoid the doublecross e_4, e_6, e_7, e. But then $b \in B(b_1, b_4]$, and (e, e_3, e_5, e_6, e_7) is a 5-edge configuration contradicting the choice of $\mathcal{P}$. □

Let $e^* = a_2b^* \in E(G)$, such that $b^* \in B(b_1, b_2)$, and $B[b^*, b_2]$ is minimal. Then by (2.1.3) and (2.1.5), $b^* \in B[b_7, b_2)$ and G has no edge from $B(b^*, b_2]$ to A.

Let $e' = a'b' \in E(G)$ with $a' \in A(a_8, a_2]$ and $b' \in B(b_6, b_2]$, such that $B[b', b_2]$ is maximal. Note that e' exists because of e^*. And $b' \in B[b_7, b^*]$ by (2.1.3).

Now, by (2.1.3), (2.1.5), and the choice of e^*, e', we have

(2.1.6) G has no edge from $B(b^*, b_2]$ to A and no edge from $B(b_1, b')$ to $A(a_8, a_2]$.

(2.1.7) G has no edge from b_1 to $A[a_1, a_8)$.

For, suppose there exists $e = ab_1 \in E(G)$ with $a \in A[a_1, a_8)$. Then, obviously, by the choice of e_8, $a \notin A(a_5, a_8)$. Hence, $a \in A[a_1, a_5]$. Since $a \neq a_1$, then let $e_0 = a_1b_0 \in E(G)$ with $b_0 \in B(b_1, b_2)$. Now, $b_0 \in B[b_7, b_2)$ to avoid the doublecross e_0, e_4, e_7, e. But then (e_0, e^*, e_5, e_4, e) is a 5-edge configuration contradicting the choice of $\mathcal{P}$. □

(2.1.8) For any edge $f_8' = a_8'b_8' \in E(G)$ with $a_8' \in A[a_5, a_2]$ and $b_8' \in B(b_6, b_2]$, G has no edge from $B(b_4, b_8')$ to $A(a_8', a_2]$.

For, otherwise, let $f_9' = a_9'b_9' \in E(G)$ with $a_9' \in A(a_8', a_2]$ and $b_9' \in B(b_4, b_8')$. Then $b_9' \notin B(b_5, b_8')$ to avoid the doublecross e_5, e_6, f_8', f_9'. So $b_9' \in B(b_4, b_5]$. Moreover, $b_9' \notin B(b_3, b_5)$; otherwise, $(e_3, e_4, f_9', e_6, e_7)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. So $b_9' = b_5$. Now, we see that $a_7 \in A[a_5, a_9')$; or else, $(e_3, e_4, f_9', e_6, e_7)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. But then $(e^*, e_7, f_9', e_3, e_8)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. □

(2.1.9) G_0 does not have a cut $\{b_3, b''\}$ or $\{b_3, b'', a''\}$ with $b'' \in B[b_6, b']$ separating $B[b_3, b'']$ from $\{a_0, b_1, b_2\}$.

For, suppose G_0 has such a cut $\{b_3, b''\}$ or $\{b_3, b'', a''\}$ with $b'' \in B[b_6, b']$. Now, let $a_9' \in A[a_1, a_2]$ such that G has an edge f_9' from a_9' to $b_9' \in B(b_3, b'')$, and subject to this, $A[a_9', a_2]$ is minimal. Obviously, by the existence of e_5, $a_9 \in A[a_5, a_2]$.

We claim that $a'_9 \notin A(a_8, a_2]$, and so by (2.1.7), G has no edge from b_1 to $A[a_1, a'_9)$. For otherwise, $b'_9 \notin B(b_3, b_7)$ to avoid the doublecross e_6, e_7, e_8, f'_9. But then $b'_9 \in B[b_7, b')$, and f'_9 contradicts the choice of e'.

By (2.1.3) and (2.1.8), G has no edge from $B(b'', b_2]$ to $A[a_1, a_9)$. Thus, $\{a_1, b_3, b'', a_9\}$ or $\{a_1, b_3, b'', a'', a_9\}$ is a cut in G separating $V(A[a_1, a_9] \cup B[b_3, b''])$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

Since (e', e_6, e_5, e_3, e_8) is a 5-edge configuration, G_0 has a cut $\{b''_1, b''_2\}$ or $\{b''_1, b''_2, a''_0\}$ (w.r.t. (e', e_6, e_5, e_3, e_8)) satisfying the conclusion of Lemma 6.0.3, such that b_1, b''_1, b''_2, b_2 occur on B in order.

Moreover, since $(e_3, e_4, e_5, e_6, e_7)$ is a 5-edge configuration, G_0 has a cut $\{b^{\#}_1, b^{\#}_2\}$ or $\{b^{\#}_1, b^{\#}_2, a^{\#}_0\}$ (w.r.t. $(e_3, e_4, e_5, e_6, e_7)$) satisfying the conclusion of Lemma 6.0.3, such that $b_1, b^{\#}_1, b^{\#}_2, b_2$ occur on B in order.

Subcase 2.1.a. Conclusions (i), or (ii), or (iii) of Lemma 6.0.3 holds for $\{b''_1, b''_2\}$ or $\{b''_1, b''_2, a''_0\}$ w.r.t. (e', e_6, e_5, e_3, e_8).

We may assume conclusion (iv) of Lemma 6.0.3 holds for $\{b^{\#}_1, b^{\#}_2\}$ w.r.t. $(e_3, e_4, e_5, e_6, e_7)$, and so $b^{\#}_1 \in B(b_4, b_5]$ and $b^{\#}_2 \in B[b_7, b_2]$. For otherwise, assume conclusions (i), or (ii), or (iii) of Lemma 6.0.3 holds for $\{b^{\#}_1, b^{\#}_2\}$ or $\{b^{\#}_1, b^{\#}_2, a^{\#}_0\}$ w.r.t. $(e_3, e_4, e_5, e_6, e_7)$. Then by the choice of $\{b'_1, b'_2\}$ with $b'_1 \neq b_1$, and by Lemma 2.0.8 and 2.0.9, we could find a cut $\{b_3, b''\}$ or $\{b_3, b'', a''\}$ with $b'' \in B[b_6, b']$ in G_0, which separates $B[b_3, b'']$ from $\{a_0, b_1, b_2\}$, a contradiction to (2.1.9).

Now, suppose conclusion (i) of Lemma 6.0.3 holds for $\{b''_1, b''_2, a''_0\}$ w.r.t. (e', e_6, e_5, e_3, e_8). Then $b''_2 \in B[b_6, b_7)$ by $b'_1 \neq b_1$ and the choice of $\{b'_1, b'_2\}$. Moreover, by Lemma 2.0.9, $b^{\#}_2 = b_2$, and $b^{\#}_1, b''_2, b_2, a_0$ are on a common finite face of G_0. Let $a'_8 \in A[a_1, a_2]$ such that G has an edge f'_8 from $B(b''_2, b_2]$ to a'_8, and $A[a'_8, a_2]$ is maximal. Now, by (2.1.3), (2.1.4), and (2.1.8), G has a separation (H_1, H_2), such that $V(H_1 \cap H_2) = \{b_1, b_2, b_4, b''_2, a'_8\}$, $\{a_0, a_1, b_1, b_2\} \subseteq V(H_1)$, and $V(A[a'_8, a_2] \cup B[b''_2, b_2]) \subseteq V(H_2)$, a contradiction.

Suppose conclusion (ii) of Lemma 6.0.3 holds for $\{b''_1, b''_2\}$ w.r.t. (e', e_6, e_5, e_3, e_8). So

$b_1'' = b_1$ and $b_2'' \in B[b_6, b']$. Then by Lemma 2.0.9, $\{b_1, b_2^{\#}\}$ is a cut in G_0 separating $B[b_1, b_2^{\#}]$ from $\{b_1, b_2, a_0\}$, which contradicts the choice of $\{b_1', b_2'\}$ (by $b_1' \neq b_1$).

So we may assume conclusion (iii) of Lemma 6.0.3 holds for $\{b_1'', b_2''\}$ w.r.t. (e', e_6, e_5, e_3, e_8). Now, $b_1'' \in B(b_1, b_3]$ and $b_2'' \in B[b_6, b']$. Then by Lemma 2.0.9, $\{b_1'', b_2^{\#}\}$ is a cut in G_0 separating $B[b_1'', b_2^{\#}]$ from $\{b_1, b_2, a_0\}$. Now, let $a_9' \in A[a_4, a_2]$, such that G has an edge f_9' from a_9' to $b_9' \in B[b_4, b_2^{\#})$, and subject to this, $A[a_9', a_2]$ is minimal.

We see that G has no edge from $B(b_2^{\#}, b_2]$ to $A[a_1, a_9')$. For, otherwise, let $f_8' = a_8'b_8' \in E(G)$ with $a_8' \in A[a_1, a_9')$ and $b_8' \in B(b_2^{\#}, b_2]$. Since $B(b_2^{\#}, b_2] \subseteq B(b_7, b_2]$, then $a_8' \notin A[a_5, a_4)$; or else, $(e_3, e_4, e_5, e_6, e_8')$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. By (2.1.3), $a_8' \in A[a_4, a_2]$. Now, by (2.1.8), $b_9' = b_4$. But now, $(e_3, f_9', e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

Thus, by (2.1.4), G has a separation (H_1, H_2), such that $V(H_1 \cap H_2) = \{b_1, b_1'', b_2^{\#}, a_9'\}$, $\{a_0, a_2, b_1, b_2\} \subseteq V(H_1)$, and $V(A[a_1, a_9'] \cup B[b_1'', b_2^{\#}]) \subseteq V(H_2)$, a contradiction.

Subcase 2.1.b. Conclusion (iv) of Lemma 6.0.3 holds for $\{b_1'', b_2''\}$ w.r.t. (e', e_6, e_5, e_3, e_8).

Then $b_2'' \in B[b_5, b_6)$, $b_1'' = b_1$, and $\{b_1, b_2''\}$ is a cut in G_0 separating $B[b_1, b_2'']$ from $\{a_0, b_1, b_2\}$. By Lemma 2.0.9, the choice of $\{b_1', b_2'\}$, and $b_1' \neq b_1$, we have $b_2' = b_2$, $b_1' \in B(b_1, b_3]$, and b_1', b_2'' are cut vertices of G_0 separating b_1 from $\{a_0, b_2\}$. So $\alpha(A, B) \leq 1$.

We may assume $b^* = b_7$. For, suppose $b^* \neq b_7$, then $b^* \in B(b_7, b_2]$. We first see that there does not exist a vertex $u \in B[b^*, b_2)$, such that b_2'', u are incident with a common finite face of G_0; or else, $\{b_1'', b_2'', u\}$ is a 3-cut in G_0 separating $B[b_1'', u]$ from $\{a_0, b_1, b_2\}$, a contradiction to the choice of $\{b_1', b_2'\}$. Then we claim that $G_0 - B[b_1, b_2''] - B[b^*, b_2)$ has disjoint paths B_2, A_0 from b_2, a_0 to b_7, b_6, respectively. For otherwise, since *Subcase* 2.1.a. does not hold, then, combined with the planar structure of G_0 and the choice of $\{b_1', b_2'\}$, there exist $u_0 \in V(G_0)$, $u_2 \in B[b^*, b_2)$, and a separation (H_1, H_2) in G_0, such that $V(H_1 \cap H_2) = \{b_2'', u_0, u_2\}$, $V(B[b_1'', b_2'') \cup B(b_2'', u_2)) \subseteq V(H_1 - H_2)$, and $\{a_0, b_2\} \subseteq V(H_2 - H_1)$. By (2.1.6), $\{b_2'', u_0', u_2\}$ is a cut in G separating $\{a_0, b_2\}$ from $\{a_1, a_2, b_1\}$, a contradiction. Hence, B_2, A_0 exist. Now, let $A' := A[a_1, a_6] \cup e_6 \cup A_0$ and $B' := B[b_1, b_5] \cup e_5 \cup A[a_5, a_7] \cup$

$e_7 \cup B_2$. Then the existence of A', B', $e_8 \cup A[a_8, a_2]$, and $e^* \cup B[b', b_2]$ implies $\alpha(A, B) = 2$ (by Lemma 3.0.1), a contradiction.

Now, by (2.1.5) and (2.1.6), G has a separation (H_1, H_2), such that $V(H_1 \cap H_2) = \{b_1, b_7, a_4\}$, $\{a_0, a_1, b_1, b_2\} \subseteq V(H_1)$, and $V(A(a_4, a_2]) \subseteq V(H_2 - H_1)$, a contradiction.

Subcase 2.2. $a_8 \in A(a_1^*, a_5]$.

Now, we may assume G has no edge from $B[b_1, t_1)$ to $A(a_5, a_2^*)$. Choose e_8 so that $A[a_8, a_5]$ is minimal, and subject to this, $B[b_8, b_1']$ is minimal. Then G has no edge from $B[b_1, b_4] \cap B[b_1, b_1')$ to $A(a_8, a_2^*)$.

(2.2.1) G has no cross from $B[b_1, b_4]$ to $A[a_1, a_5]$, and so $b_8 \in B[b_3, b_4]$.

For, suppose G has a cross from $B[b_1, b_4]$ to $A[a_1, a_5]$. Then such a cross together with e_4, e_5 forms a doublecross, a contradiction. Now, by the choice of e_8, we may assume $b_8 \in B[b_3, b_4]$. □

(2.2.2) G has no edge from $B(b_8, b_7)$ to $A[a_1, a_8) \cap A[a_1, a_7)$, and so if $a_8 \in A[a_1, a_7]$, then $b_1^* \in B[b_7, b_2)$.

For otherwise, such an edge together with e_4, e_7, e_8 (when $b_8 \neq b_4$) or f_4, e_7, e_8 (when $(iv)(b)$ occurs with $b_8 = b_4$) forms a doublecross, a contradiction. □

(2.2.3) $a_7 \in A[a_1, a_5]$.

For, suppose $a_7 \in A(a_5, a_2]$. Then $b_1^* \in B[b_7, b_2)$ by (2.2.2). So $b_7 \neq b_2$ (by $b_1^* \neq b_2$). Now, we may assume $(iv)(b)$ occurs with $b_8 = b_4$; otherwise, $b_8 \in B[b_1, b_4)$ and $(f_1, e_7, e_5, e_4, e_8)$ contradicts the choice of $\mathcal{P}$. But then $(f_1, e_7, e_5, f_4, e_8)$ is a 5-edge configuration, so by Lemma 2.0.9 and 6.0.3, G_0 has a cut contradicting the choice of $\{b_1', b_2'\}$. □

(2.2.4) G has no edge from $B(b_5, b_7)$ to $A[a_1, a_7)$, and so $a_6 \in A(a_5, a_2]$.

For, otherwise, let $e_9 = a_9b_9 \in E(G)$ with $a_9 \in A[a_1, a_7)$ and $b_9 \in B(b_5, b_7)$. Then $a_8 \in A[a_1, a_9]$ and $b_1^* \in B[b_7, b_2)$ by (2.2.2). So $b_7 \neq b_2$ (by $b_1^* \neq b_2$). Now, we may assume $(iv)(b)$ occurs with $b_8 = b_4$; otherwise, $(f_1, e_7, e_9, e_4, e_8)$ contradicts the choice of $\mathcal{P}$. But then, $(f_1, e_7, e_9, f_4, e_8)$ is a 5-edge configuration, so by Lemma 2.0.9 and 6.0.3, G_0 has a cut contradicting the choice of $\{b_1', b_2'\}$. $\Box$

(2.2.5) G has no edge from $B(b_4, b_5]$ to $A[a_1, a_5)$.

For, otherwise, let $e_9 = a_9b_9 \in E(G)$ with $a_9 \in A[a_1, a_5)$ and $b_9 \in B(b_4, b_5]$. Then $a_9 \notin A[a_7, a_5)$; otherwise, $(e_3, e_4, f_9, e_6, e_7)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. Moreover, $a_8 \in A[a_1, a_9]$ and $b_1^* \in B[b_7, b_2)$ by (2.2.2). So $b_7 \neq b_2$ (by $b_1^* \neq b_2$). Now, we may assume $(iv)(b)$ occurs with $b_8 = b_4$; otherwise, $(f_1, e_7, e_9, e_4, e_8)$ contradicts the choice of $\mathcal{P}$. But then, $b_9 = b_5$, and $(f_1, e_7, e_9, f_4, e_8)$ is a 5-edge configuration, so by Lemma 2.0.9 and 6.0.3, G_0 has a cut contradicting the choice of $\{b_1', b_2'\}$. $\Box$

(2.2.6) G has no edge from $B(b_6, b_2]$ to $A(a_5, a_2]$.

For, otherwise, let $e_9 = a_9b_9 \in E(G)$ with $a_9 \in A(a_5, a_2]$ and $b_9 \in B(b_6, b_2]$. Then $b_9 \in B[b_7, b_2]$; or else, $(e_3, e_4, e_5, e_9, e_7)$ contradicts the choice of $\mathcal{P}$.

We see that $b_9 \neq b_2$. For otherwise, $a_9 \neq a_2$. Let $e = a_2b \in E(G)$ with $b \in B(b_1, b_2)$ and $b \neq b_4$. Then $b \notin B(b_1, b_4)$; otherwise, (e_3, e, e_5, f_1, e_9) contradicts the choice of $\mathcal{P}$. So $b \in B(b_4, b_2)$. Now, e_8, e_9, f_1, e form a doublecross, a contradiction.

So $b_9 \in B[b_7, b_2)$ and $b_7 \neq b_2$. Now, we may assume $(iv)(b)$ occurs with $b_8 = b_4$; otherwise, combined with (2.2.2), $(e_9, e_7, e_5, e_4, e_8)$ (when $a_7 \in A[a_1, a_8)$) or $(e_9, f_1, e_5, e_4, e_8)$ (when $a_8 \in A[a_1, a_7]$) contradicts the choice of $\mathcal{P}$. But then, $(e_9, e_7, e_5, f_4, e_8)$ (when $a_7 \in A[a_1, a_8)$) or $(e_9, f_1, e_5, f_4, e_8)$ (when $a_8 \in A[a_1, a_7]$) is a 5-edge configuration. By Lemma 2.0.9 and 6.0.3, G_0 has a cut contradicting the choice of $\{b_1', b_2'\}$. $\Box$

Now, by (2.2.3)–(2.2.6) and by Lemma 6.0.4,

(2.2.7) G_0 does not contain a cut $\{b_1'', b_2''\}$ separating $B[b_1'', b_2'']$ from $\{a_0, b_1, b_2\}$ with $b_1'' \in B[b_1, b_4]$ and $b_2'' \in B[b_6, b_2]$.

By (2.2.7), we have

(2.2.8) Conclusions (ii) and (iii) of Lemma 6.0.5 do not hold for $\{b_1', b_2'\}$ w.r.t. $(e_3, e_4, e_5, e_6, e_7)$.

(2.2.9) G has no edge from $B[b_1, b_4)$ to $A(a_5, a_2]$.

For, suppose G has an edge e from $b \in B[b_1, b_4)$ to $a \in A(a_5, a_2]$. Then we may assume $b = b_1$. For otherwise, $b \in B(b_1, b_4)$. Now, $a \in A(a_5, a_4)$ and $b \in B(b, b_4]$; or else, (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. But then e_3, e_4, e, e_5 form a doublecross.

So $a \neq a_2$ (by $b = b_1$). Now, let $e_0 = a_2b_0 \in E(G)$ with $b_0 \in B(b_1, b_2)$. If $b_0 \in B(b_1, b_7)$, then e_0, e, e_3, e_7 form a doublecross. So, $b_0 \in B[b_7, b_2)$, contradicting (2.2.6). □

(2.2.10) G has no parallel edges from $A[a_1, a_8]$ to $B[b_4, b_2]$ and no parallel edges from $A[a_1, a_5]$ to $B[b_6, b_2]$.

For otherwise, such two parallel edges together with e_4, e_8 or e_5, e_6 form a doublecross, a contradiction. □

Let $e_7' = a_7'b_7' \in E(G)$ with $a_7' \in A[a_1, a_7]$ and $b_7' \in B[b_7, b_2]$, such that $A[a_1, a_7'] \cup B[b_7', b_2]$ is minimal.

(2.2.11) $a_7' \in A[a_1, a_8)$, and G has no edge from $B(b_7', b_2]$ to A.

For, suppose $a_7' \notin A[a_1, a_8)$. Since $a_1^* \in A[a_1, a_8)$, then by the choice of e_7', $b_1^* \in B(b_8, b_7')$, and so e_8, e_4, f_1, e_7' form a doublecross.

By (2.2.6) and (2.2.10), and by the choice of e_7', G has no edge from $B(b_7', b_2]$ to A. □

Let $e' = a'b' \in E(G)$ with $a' \in A[a_1, a_5]$ and $b' \in B[b_1, t_1)$, such that $A[a_1, a'] \cup B[b_1, b']$ is minimal.

By (2.2.1) and (2.2.9), and by the choice of e', we have

(2.2.12) e', e_8 do not form a cross, and G has no edge from $B[b_1, b')$ to A, and no edge from $B(b', b_8)$ to $A[a_1, a') \cup A(a_8, a_2]$.

(2.2.13) If conclusion (iv) of Lemma 6.0.5 holds for $\{b'_1, b'_2\}$ w.r.t. $(e_3, e_4, e_5, e_6, e_7)$, then there does not exist a 3-cut $\{b''_1, b''_2, a''_0\}$ in G_0 with $b''_1 \in B[b_1, b_4]$ and $b''_2 \in B(b_5, b_2)$, which separates $B[b''_1, b''_2]$ from $\{a_0, b_1, b_2\}$.

For, suppose conclusion (iv) of Lemma 6.0.5 holds for $\{b'_1, b'_2\}$ w.r.t. $(e_3, e_4, e_5, e_6, e_7)$, and (2.2.13) fails. Then $b'_1 \in B(b_4, b_5]$, $b'_2 \in B[b_7, b_2]$, and G has no edge from $B(b_4, b'_1)$ to $A - a_4$. Now, by the choice of $\{b'_1, b'_2\}$ and by Lemma 2.0.9, $b''_1 = b_1, b''_2 \in B(b_5, b_7)$, $a''_0 = a_0$, $b'_2 = b_2$, and $\alpha(A, B) \leq 1$.

By the choice of $\{b'_1, b'_2\}$, and by planar structure of G_0, we may assume $G_0 - a_0 - B[b'_7, b_2)$ contains a path B_2 from b_2 to b''_2.

Let $e'_4 = a_4 b'_4 \in E(G)$ with $b'_4 \in B[b_4, b'_1)$ such that $B[b'_4, b'_1]$ is minimal. Since $b_8 \in B[b_1, t_1)$, then $b_8 \neq b'_4$.

We claim that if $b'_4 \neq b_4$, then G has no edge from $B[b_1, b'_4)$ to $A(a_5, a_2] - a_4$. For, suppose $b'_4 \in B(b_4, b'_1)$ and G has an edge e from $b \in B[b_1, b'_4)$ to $a \in A(a_5, a_2] - a_4$. Now, by (2.2.9), $b \in B[b_4, b'_4)$. By (iv) of Lemma 6.0.5, $b \notin B(b_4, b'_4)$. So $b = b_4$. By the choice of $\mathcal{P}$, $a \in A(a_5, a_4)$. Now, let $e_0 = ab_0 \in E(G)$ with $b_0 \in B[b_1, b_2]$ and $b_0 \notin \{b_4, b_5\}$. By (2.2.9), $b_0 \notin B[b_1, b_4)$. Moreover, $b_0 \notin B(b_5, b_2]$ to avoid the doublecross e, e_0, e'_4, e_5. So $b_0 \in B(b_4, b_5)$. But then e, e_6, e'_4, e_5 form a doublecross (when $a_6 \in A(a_5, a_4)$), or $(e_3, e_4, e_0, e_6, e_7)$ contradicts the choice of $\mathcal{P}$ (when $a_6 \in A[a_4, a_2]$).

By the choice of e_8, (2.2.1), (2.2.9), (iv) of Lemma 6.0.5, and our previous claim, if $b'_4 = b_4$, then G has no edge from $B(b_8, b'_4)$ to A; if $b'_4 \neq b_4$, then G has no edge from $B(b_8, b'_4)$ to $A - a_4$.

Now, we see that $e' \cap e_8 = \emptyset$. For, suppose there exists a vertex $v \in e' \cap e_8$. Then, by (2.2.12), (iv) of Lemma 6.0.5, and our previous analysis, $\{b_1, v, b_4, b'_1, b_2\}$ (when $b'_4 = b_4$) or $\{b_1, v, a_4, b'_1, b_2\}$ (when $b'_4 \neq b_4$) is a cut in G separating a_0 from $V(A)$, a contradiction.

We may assume $G_0 - B(b_1, b'] - B[b'_1, b_2]$ contains disjoint paths B_1, A_0 from b_1, a_0 to b_8, b'_4, respectively. For, suppose not. There exists a vertex $v \in V(G_0)$, such that v is a cut vertex in $G_0 - B(b_1, b'] - B[b'_1, b_2]$ separating b_1, a_0 from b_8, b'_4. We see that $v \notin B[b', b_8]$;

otherwise, v and b_1' are incident with a common finite face of G_0, and so $\{v, b_1', b_2'\}$ is a 3-cut in G_0 separating $B[v, b_2']$ from $\{a_0, b_1, b_2\}$, a contradiction to the choice of $\{b_1', b_2'\}$. Moreover, $v \notin B[b_4', b_1']$. For otherwise, there exists a vertex $v_1 \in B(b_1, b']$, such that v_1, v are incident with a common finite face of G_0. By (2.2.12), (iv) of Lemma 6.0.5, and the choice of e_4', $\{v_1, v, b_1'\}$ is a cut in G separating $\{a_0, b_1\}$ from $\{a_1, a_2, b_2\}$, a contradiction. Now, we may assume $v \notin V(B)$, and so there exists a vertex $v_1 \in B(b_1, b']$, such that v_1, v are incident with a common finite face of G_0, and v, b_1' are incident with a common finite face of G_0. But then, combined with (2.2.12), $\{v_1, v, b_1'\}$ is still a cut in G separating $\{a_0, b_1\}$ from $\{a_1, a_2, b_2\}$, a contradiction.

Now, combined with Lemma 3.0.1, the path $B_1 \cup e_8 \cup A[a_8, a_5] \cup e_5 \cup B[b_5, b_2''] \cup B_2$ from b_1 to b_2, the path $A[a_4, a_2] \cup e_4' \cup A_0$ from a_2 to a_0, the path $A[a_1, a'] \cup e' \cup B[b_1, b']$ from a_1 to b_1, and the path $A[a_1, a_7'] \cup e_7' \cup B[b_7', b_2]$ from a_1 to b_2 show that $\alpha(A, B) = 2$, a contradiction. □

(2.2.14) Conclusion (i) of Lemma 6.0.5 holds for $\{b_1', b_2'\}$ w.r.t. $(e_3, e_4, e_5, e_6, e_7)$, $b_8 \neq b_4$, and G has no edge from $B[b_1, b_1')$ to $A(a_8, a_2]$.

For, suppose conclusion (i) of Lemma 6.0.5 does not hold for $\{b_1', b_2'\}$. By (2.2.8), conclusion (iv) of Lemma 6.0.5 holds for $\{b_1', b_2'\}$ w.r.t. $(e_3, e_4, e_5, e_6, e_7)$. So $b_1' \in B(b_4, b_5]$ and $b_2' \in B[b_7, b_2]$. By (2.2.1) and (2.2.5), $b_1^* \in B(b_5, b_2)$. Hence, $(f_1, e_6, e_5, e_4, e_8)$ (when (iv)(a) occurs) or $(f_1, e_6, e_5, f_4, e_8)$ (when (iv)(b) occurs) is a 5-edge configuration. However, by Lemma 2.0.9 and 6.0.3, G_0 has a cut contradicting (2.2.13) or the choice of $\{b_1', b_2'\}$.

Hence, conclusion (i) of Lemma 6.0.5 holds for $\{b_1', b_2'\}$ w.r.t. $(e_3, e_4, e_5, e_6, e_7)$, which implies that $b_1' \in B[b_1, b_4]$. Since $b_8 \in B[b_1, b_1')$, then $b_8 \neq b_4$. By (2.2.9), G has no edge from $B[b_1, b_1')$ to $A(a_5, a_2]$. Now, by the choice of e_8, G has no edge from $B[b_1, b_1')$ to $A(a_8, a_2]$. □

(2.2.15) G has no edge from $B(b_8, b_6)$ to $A[a_1, a_8)$, and so $(f_1, e_6, e_5, e_4, e_8)$ is a 5-edge configuration with $b_1^* \in B[b_6, b_2)$.

We first claim that G has no edge from $B(b_8, b_6)$ to $A[a_1, a_8)$. For, suppose there exists $e = ab \in E(G)$ with $b \in B(b_8, b_6)$ and $a \in A[a_1, a_8)$. We may assume $a_7 \in A(a_1, a]$ to avoid the doublecross e_4, e_7, e_8, e. But now, since $a_3 \in A[a_1, a_7)$, then, combined with (2.2.1), $b_3 \in B(b_1, b_8]$, and so (e_3, e_8, e, e_6, e_7) contradicts the choice of $\mathcal{P}$.

Now, by our claim, $b_1^* \notin B(b_8, b_6)$, and so $(f_1, e_6, e_5, e_4, e_8)$ is a 5-edge configuration with $b_1^* \in B[b_6, b_2)$. □

We choose f_1 so that $B[b_6, b_1^*]$ is minimal. Moreover, we let $e_5' = a_5'b_5' \in E(G)$ with $a_5' \in A(a_1^*, a_6)$ and $b_5' \in B[b_5, b_6)$ so that $B[b_5', b_6]$ is minimal. Now, since $(f_1, e_6, e_5', e_4, e_8)$ is a 5-edge configuration, G_0 has a cut $\{b_1^\#, b_2^\#\}$ or $\{b_1^\#, b_2^\#, a_0^\#\}$ (w.r.t. $(f_1, e_6, e_5', e_4, e_8)$) satisfying the conclusion of Lemma 6.0.3, such that $b_1, b_1^\#, b_2^\#, b_2$ occur on B in order.

By (2.2.7), we have

(2.2.16) Conclusions (ii) and (iii) of Lemma 6.0.3 do not hold for $\{b_1^\#, b_2^\#\}$ w.r.t. $(f_1, e_6, e_5', e_4, e_8)$.

(2.2.17) Conclusion (i) of Lemma 6.0.3 holds for $\{a_0^\#, b_1^\#, b_2^\#\}$ w.r.t. $(f_1, e_6, e_5', e_4, e_8)$.

For, otherwise, by (2.2.16), conclusion (iv) of Lemma 6.0.3 holds for $\{b_1^\#, b_2^\#\}$ w.r.t. $(f_1, e_6, e_5', e_4, e_8)$. So $b_1^\# \in B[b_1, b_8]$ and $b_2^\# \in B[b_5', b_6)$. Then by Lemma 2.0.9, and by the choice of $\{b_1', b_2'\}$, we have $b_1^\# = b_1, b_2' = b_2$, $a_0 = a_0'$, and $\alpha(A, B) \leq 1$. We further choose $\{b_1^\#, b_2^\#\}$ so that $B[b_2^\#, b_2]$ is minimal.

By the choice of $\{b_1', b_2'\}$, and by planar structure of G_0, we may assume $G_0 - a_0 - B(b_1, b_1')$ contains a path B_1 from b_1 to b_1'.

We let $e_6' = a_6'b_6' \in E(G)$ with $a_6' \in A(a_5, a_2]$ and $b_6' \in B(b_2^\#, b_6]$, such that $A[b_6', b_2]$ is maximal.

G has no edge from $B(b_5', b_6')$ to A. For, suppose G has an edge e from $b \in B(b_5', b_6')$ to $a \in A$. Then by the choice of e_6', $a \in A[a_1, a_5]$. By the choice of e_5', $a \notin A(a_1^*, a_6)$. So $a \in A[a_1, a_1^*]$, which contradicts (2.2.15).

Let A_0 be the path from a_0 to b_6' on the boundary of $G_0 - B[b_1, b_2^\#]$ without going through b_2. Since (2.2.17) fails, then combined with the choice of $\{b_1^\#, b_2^\#\}$, we may assume

$A_0 \cap B(b_6, b_2] = \emptyset$.

We claim that G has an edge e_7'' from $a_7'' \in A(a_7', a_8)$ to $b_7'' \in B(b_6, b_7')$. In fact, we see that G has an edge e from $a \in A[a_1, a_8)$ to $b \in B[b_1', b_2] - \{b_6\}$, such that $e \cap e_7' = \emptyset$; otherwise, by (2.2.1) and (2.2.10), G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{b_1, b_1', a_8, b_6, u, a_1\}$ with $u \in \{a_7', b_7'\}$, $V(A[a_1, a_8] \cup B[b_1, b_1']) \subseteq V(G_1)$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_2)$, and $(G_1, b_1, b_1', a_8, b_6, u, a_1)$ is planar, a contradiction to Lemma 2.0.3. Now, by (2.2.15), $b \notin B(b_8, b_6)$, and so $b \in B(b_6, b_2]$. Finally, by (2.2.10) and the choice of e_7', $a \in A(a_7', a_8)$ and $b \in B(b_6, b_7')$, which finishes our claim.

We further choose e_7'' with $a_7'' \in A(a_7', a_8)$ and $b_7'' \in B(b_6, b_7')$ so that $A[a_1, a_7'']$ is maximal. Now, we may assume $a_7'' \in A(a', a_8)$. For otherwise, $a_7'' \in A[a_1, a']$. By (2.2.10), (2.2.15), and the choice of e_7'', $\{b_1, b_1', a', a_8, b_6\}$ is a cut in G separating $V(A[a', a_8] \cup B[b_1, b_1'])$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

We may also assume $G_0 - A_0 - B[b_7', b_2)$ contains a path B_2 from b_2 to b_7''. For otherwise, $b_7' \neq b_2$, and there exist a vertex $v_1 \in A_0$ and a vertex $v_2 \in B[b_7', b_2)$, such that v_1, v_2 are incident with a common finite face in G_0. If $v_1 = a_0$, then $\{v_1, v_2, b_2\}$ is a cut in G separating $N_G(b_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction; if $v_1 \neq a$, then combined with (2.2.11), $\{b_1, b_2^{\#}, v_1, v_2, b_2\}$ is a cut in G separating a_0 from $\{a_1, a_2\}$, a contradiction.

Now, combined with Lemma 3.0.1, the path $B_1 \cup B[b_1', b_5] \cup e_5 \cup A[a_7'', a_5] \cup e_7'' \cup B_2$ from b_1 to b_2, the path $A[a_6', a_2] \cup e_6' \cup A_0$ from a_2 to a_0, the path $A[a_1, a'] \cup e' \cup B[b_1, b']$ from a_1 to b_1, and the path $A[a_1, a_7'] \cup e_7' \cup B[b_7', b_2]$ from a_1 to b_2 show that $\alpha(A, B) = 2$, a contradiction. $\square$

Now, by (2.2.17), $b_1^{\#} \in B[b_1, b_8]$ and $b_2^{\#} \in B[b_6, b_1^*]$. Moreover, we choose $\{b_1^{\#}, b_2^{\#}\}$ so that $B[b_1^{\#}, b_2^{\#}]$ is maximal. By (2.2.7), G_0 contains a path from a_0 to $B(b_4, b_6)$, internally disjoint from B. Then by Lemma 2.0.8, and by the choice of $\{b_1', b_2'\}$, we have $b_1^{\#} = b_1, b_2' = b_2$, and one of the following holds:

(N1) $a_0 = a_0' = a_0^{\#}$, and so $c(A, B) \geq 2$;

(N2) $a_0^{\#} = a_0$, $b_2^{\#}$ is a cut vertex of G_0 separating b_2 from $\{a_0, b_1\}$, $a'_0, a_0^{\#}, b_2^{\#}, b'_2$ are incident with a common finite face of G_0, and so $\alpha(A, B) \leq 1$;

(N3) $a'_0 = a_0, b'_1$ is a cut vertex of G_0 separating b_1 from $\{a_0, b_2\}$, $a'_0, a_0^{\#}, b_1^{\#}, b'_1$ are incident with a common finite face of G_0, and so $\alpha(A, B) \leq 1$.

Obviously, by (N1)–(N3), there exists a vertex $a_0^* \in \{a'_0, a_0^{\#}\}$, such that $\{b'_1, b_2^{\#}, a_0^*\}$ is a 3-cut in G_0 separating $B[b'_1, b_2^{\#}]$ from $\{a_0, b_1, b_2\}$. We let $e_9 = a_9 b_9 \in E(G)$ with $b_9 \in B(b'_1, b_2^{\#})$ and $a_9 \in A[a_1, a_2]$, such that $A[a_1, a_9]$ is minimal.

(2.2.18) There exists $e'_9 = a'_9 b'_9 \in E(G)$ with $a'_9 \in A(a_9, a_2]$ and $b'_9 \in B[b_1, b'_1) \cup B(b_2^{\#}, b_2]$.

For otherwise, $\{a_0^*, b'_1, b_2^{\#}, a_9, a_2\}$ is a cut in G separating $A[a_9, a_2] \cup B[b'_1, b_2^{\#}]$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

(2.2.19) $b_9 \in B(b'_1, b_4]$, $a_9 \in A[a_8, a_5)$, $a'_9 \in A(a_9, a_5]$, and $b'_9 \in B(b_2^{\#}, b_7]$.

We first prove that $a_9 \notin A[a_1, a_8)$. For, suppose $a_9 \in A[a_1, a_8)$. Then by (2.2.15), $b_9 \notin B(b_8, b_6)$, and so $b_9 \in B[b_6, b_2^{\#})$, a contradiction to the choice of f_1.

We claim that $b'_9 \in B(b_2^{\#}, b_2]$. For, suppose, $b'_9 \in B[b_1, b'_1)$. By (2.2.9), $a'_9 \notin A(a_5, a_2]$, and so $a'_9 \in A(a_9, a_5]$, a contradiction to the choice of e_8.

By (2.2.6), $a'_9 \notin A(a_5, a_2]$, and so $a'_9 \in A(a_9, a_5]$. Furthermore, we have $b'_9 \in B(b_2^{\#}, b_7]$; or else, $(e_3, e_4, e_5, e_6, e'_9)$ contradicts the choice of $\mathcal{P}$.

Now, since $a'_9 \in A(a_9, a_5]$, then $a_9 \neq a_5$, which implies that $a_9 \in A[a_8, a_5)$.

Finally, $b_9 \in B(b'_1, b_4]$. First, $b_9 \notin B(b_5, b_2^{\#})$ to avoid the doublecross e'_9, e_5, e_6, e_9. By (2.2.5), $b_9 \notin B(b_4, b_5]$. So $b_9 \in B(b'_1, b_4]$. □

Now, we choose e'_9 so that $B[b_2^{\#}, b'_9]$ is minimal. Since $a'_9 \in A(a_9, a_5]$, then $a_5 \neq a_9$.

(2.2.20) (N1) does not hold.

For, suppose (N1) holds. By the choice of $\{b_1', b_2'\}$, and by planar structure of G_0, $G_0 - B(b_1, b_1') - a_0$ contains a path B_1 from b_1 to b_1'. Moreover, by the choice of $\{b_1^{\#}, b_2^{\#}\}$, and by planar structure of G_0, $G_0 - B(b_2^{\#}, b_2) - a_0$ contains a path B_2 from $b_2^{\#}$ to b_2.

We may assume G has two disjoint edges f_8, f_9 from $a_8^*, a_9^* \in A(a_1, a_8)$ to $b_8^*, b_9^* \in B(b_1', b_2]$, respectively. For otherwise, there exist a vertex $v \in V(G)$ and a separation (G_1, G_2) in G, such that $V(G_1 \cap G_2) = \{b_1', a_0, b_1, a_1, v, a_8\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(A(a_1, a_8) \cup B(b_1, b_1')) \subseteq V(G_2)$, and $(G_2, b_1', a_0, b_1, a_1, v, a_8)$ is planar, a contradiction to Lemma 2.0.3.

By (2.2.15), $b_8^*, b_9^* \in B[b_6, b_2]$. Moreover, by (2.2.10), f_8, f_9 form a cross. So we may assume a_1, a_8^*, a_9^*, a_2 occur on A in order, and b_1, b_9^*, b_8^*, b_2 occur on B in order. We further choose f_8, f_9 with $a_8^*, a_9^* \in A(a_1, a_8)$ and $b_8^*, b_9^* \in B[b_6, b_2]$ so that $A[a_8^*, a_9^*]$ is maximal. By the existence of e_9' and (2.2.10), we may assume $b_8^* \in B(b_2^{\#}, b_2]$.

We claim that G has an edge f_5 from $b_5^* \in B[b_1, b_1')$ to $a_5^* \in A(a_1, a_9^*)$. For otherwise, all edges from $B[b_1, b_1')$ will end in $\{a_1\} \cup V(A[a_9^*, a_8])$. By the choice of f_8, f_9, G has no edge from $A(a_9^*, a_8)$ to $B(b_8, b_2]$. Hence, G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{b_1', a_0, b_1, a_1, a_9^*, a_8\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(A(a_9^*, a_8) \cup B(b_1, b_1')) \subseteq V(G_2)$, and $(G_2, b_1', a_0, b_1, a_1, a_9^*, a_8)$ is planar. By Lemma 2.0.3, $|V(G_2 - G_1)| = 1$. So $V(G_2 - G_1) = \{b_8\}$, and G has edges from b_8 to $b_1', a_0, b_1, a_1, a_9^*, a_8$, respectively. But then, b_1 has degree 1 in G, a contradiction.

By (2.2.7), there exists a path A_0 from a_0 to $B(b_4, b_6)$ in G_0, internally disjoint from B.

Now, combined with Lemma 3.0.1, the path $B_1 \cup B[b_1', b_9] \cup e_9 \cup A[a_9^*, a_9] \cup f_9 \cup B[b_9^*, b_2^{\#}] \cup B_2$ from b_1 to b_2, the path $B[b_1, b_5^*] \cup f_5 \cup A[a_5^*, a_8^*] \cup f_8 \cup B[b_8^*, b_2]$ from b_1 to b_2, and the path $A_0 \cup B(b_4, b_6) \cup e_5 \cup A[a_5, a_2]$ from a_0 to a_2 show that $\alpha(A, B) = 2$ and $c(A, B) = 0$, a contradiction. □

(2.2.21) (N2) does not hold.

For, suppose (N2) holds. We may assume G has an edge e_7'' from $a_7'' \in A[a_1, a_8)$ to $b_7'' \in B(b_1', b_2]$, such that $e_7'' \cap e_7' = \emptyset$. For otherwise, by (2.2.1), (2.2.10) and (2.2.15), G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{v, a_8, b'_1, a'_0\}$ with $v \in \{a'_7, b'_7\}$, $a_0, a_1, b_1 \in V(G_2)$, $|V(G_2 - G_1)| \geq 4$, $a_2, b_2 \in V(G_1)$, and $(G_2, a_0, b_1, a_1, v, a_8, b'_1, a'_0)$ is planar. Now, it contradicts Lemma 2.0.3 (when $v = a'_7 = a_1$) or Lemma 2.0.4 (when $v \neq a_1$).

By (2.2.10) and (2.2.15), $a''_7 \in A(a'_7, a_8)$ and $b''_7 \in B[b_6, b'_7)$. Now, we choose e''_7 so that $A[a_1, a''_7]$ is maximal. Then we may assume $a''_7 \in A(a', a_8)$. For otherwise, $a''_7 \in A[a_1, a']$, and so G has no edge from $A(a', a_8)$ to $B(b'_1, b_2]$ by the choice of e''_7. But then, G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{a', a_8, b'_1, a'_0, a_0, b_1\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, and $(G_2, a', a_8, b'_1, a'_0, a_0, b_1)$ is planar, a contradiction to Lemma 2.0.3.

By the choice of $\{a_0^{\#}, b_1^{\#}, b_2^{\#}\}$, and by planar structure of G_0, we may assume $G_0 - B[b'_7, b_2)$ contains a path B_2 from b_2 to $b_2^{\#}$.

Now, let A_0 be the path from a_0 to $B(b_4, b_6)$ in G_0, internally disjoint from B. Moreover, we further choose A_0 such that $A_0[a_0, a'_0]$ is on the boundary of G_0 without going through b_1.

We claim that $G_0 - B(b_1, b'] - A_0$ contains a path B_1 from b_1 to b'_1. For otherwise, $b'_1 \neq b_1$, and there exist a vertex $v_1 \in A_0[a_0, a'_0]$ and a vertex $v_2 \in B(b_1, b']$, such that v_1, v_2 are incident with a common finite face of G_0. Now, combined with (2.2.12), if $v_1 \neq a_0$, then $\{b_1, v_1, v_2, b_2\}$ is a cut in G separating a_0 from $\{a_1, a_2\}$, a contradiction; if $v_1 = a_0$, then $\{v_1, v_2, b_1\}$ is a cut in G separating $N_G(b_1)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Now, combined with Lemma 3.0.1, the path $B_1 \cup B[b'_1, b_9] \cup e_9 \cup A[a''_7, a_9] \cup e''_7 \cup B[b''_7, b_2^{\#}] \cup B_2$ from b_1 to b_2, the path $A_0 \cup B(b_4, b_6) \cup e_5 \cup A[a_5, a_2]$ from a_0 to a_2, the path $A[a_1, a'] \cup e' \cup B[b_1, b']$ from a_1 to b_1, and the path $A[a_1, a'_7] \cup e'_7 \cup B[b'_7, b_2]$ show that $\alpha(A, B) = 2$, a contradiction. $\square$

Hence, (N3) holds. We may assume G has an edge e''_7 from $a''_7 \in A(a', a_8)$ to $b''_7 \in B(b'_1, b_2]$, such that $e''_7 \cap e'_7 = \emptyset$. For otherwise, by (2.2.10) and (2.2.15), G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{v, a', a_8, b_1, b'_1\}$ with $v \in \{a'_7, b'_7\}$, $V(A[a', a_8] \cup B[b_1, b'_1]) \subseteq V(G_1)$, and $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_2)$, a contradiction.

By (2.2.10) and (2.2.15), we may also assume $a''_7 \in A(a'_7, a_8)$ and $b''_7 \in B[b_6, b'_7)$. By the choice of $\{a_0', b_1', b_2'\}$, and by planar structure of G_0, we may assume $G_0 - B(b_1, b']$ contains a path B_1 from b_1 to b_1'.

Now, let A_0 be the path from a_0 to $B(b_4, b_6)$ in G_0, internally disjoint from B. Moreover, we further choose A_0 such that $A_0[a_0, a_0^{\#}]$ is on the boundary of G_0 without going through b_2.

We claim that $G_0 - B[b_7', b_2) - A_0$ contains a path B_2 from b_2 to $b_2^{\#}$. For otherwise, $b_7' \neq b_2$, and there exist a vertex $v_1 \in A_0[a_0, a_0^{\#}]$ and a vertex $v_2 \in B[b_7', b_2)$, such that v_1, v_2 are incident with a common finite face of G_0. Now, combined with (2.2.11), if $v_1 \neq a_0$, then $\{b_1, v_1, v_2, b_2\}$ is a cut in G separating a_0 from $\{a_1, a_2\}$, a contradiction; if $v_1 = a_0$, then $\{v_1, v_2, b_2\}$ is a cut in G separating $N_G(b_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Now, combined with Lemma 3.0.1, the path $B_1 \cup B[b_1', b_9] \cup e_9 \cup A[a_7'', a_9] \cup e_7'' \cup B[b_7'', b_2^{\#}] \cup B_2$ from b_1 to b_2, the path $A_0 \cup B(b_4, b_6) \cup e_5 \cup A[a_5, a_2]$ from a_0 to a_2, the path $A[a_1, a'] \cup e' \cup B[b_1, b']$ from a_1 to b_1, and the path $A[a_1, a_7'] \cup e_7' \cup B[b_7', b_2]$ show that $\alpha(A, B) = 2$, a contradiction. □

CHAPTER 7

FUTURE WORK

7.0.1 A characterization of two-three linked graphs

In fact, Robertson and Seymour asked for a characterization of two-three linked graphs. Here, we believe we have such a characterization, although it is quite complicated (even to state) and its proof is longer.

We say that $(G, a_0, a_1, a_2, b_1, b_2)$ is *reducible*, if one of the following holds:

(R1) G has an edge e with one end in $\{a_0, a_1, a_2\}$ and one end in $\{b_1, b_2\}$.

(R2) There exists a separation (G_1, G_2) in G of order at most 1.

(R3) There exists a separation (G_1, G_2) in G of order 2, satisfying one of the following properties:

  (a) $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$ and $V(G_2 - G_1) \neq \emptyset$; or

  (b) $|V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}| = 1$ and $|E(G_2)| \geq 3$; or

  (c) for some $i \in \{0, 1, 2\}$ and some $j \in \{1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2\}$, $a_i, b_j \in V(G_2 - G_1)$, $\{a_0, a_1, a_2, b_1, b_2\} - \{a_i, b_j\} \subseteq V(G_1)$, and $(G_2, a_i, b_j, c_2, c_1)$ is planar; or

  (d) for some $j \in \{1, 2\}$ and some permutation π of $\{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2\}$, $a_{\pi(0)}, a_{\pi(1)}, b_j \in V(G_2 - G_1)$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, and $(G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_2, c_1)$ is planar; or

  (e) for some $i \in \{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2\}$, $a_i, b_1, b_2 \in V(G_2 - G_1)$, $\{a_0, a_1, a_2, b_1, b_2\} - \{a_i, b_1, b_2\} \subseteq V(G_1)$, and $(G_2, b_1, a_i, b_2, c_2, c_1)$ is planar.

(R4) There exists a separation (G_1, G_2) in G of order 3, satisfying one of the following properties:

(a) $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$ and $V(G_2 - G_1) \neq \emptyset$; or

(b) $V(G_1 \cap G_2) = \{c_1, c_2, c_3\}$, $\{d\} = \{a_0, a_1, a_2, b_1, b_2\} \cap V(G_2 - G_1)$, (G_2, d, c_3, c_2, c_1) is planar, and $|V(G_2 - G_1)| \geq 2$; or

(c) for some $i \in \{0, 1, 2\}$ and some $j \in \{1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3\}$, $a_i, b_j \in V(G_2 - G_1)$, $\{a_0, a_1, a_2, b_1, b_2\} - \{a_i, b_j\} \subseteq V(G_1)$, $(G_2, a_i, b_j, c_1, c_2, c_3)$ is planar, and $|V(G_2 - G_1)| \geq 3$; or

(d) for some permutation π of $\{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3\}$, $a_{\pi(0)}, a_{\pi(1)}, b_j \in V(G_2 - G_1)$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, and $(G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_3, c_2, c_1)$ is planar; or

(e) for some $i \in \{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3\}$, $b_1, a_i, b_2 \in V(G_2 - G_1)$, $\{a_0, a_1, a_2\} - \{a_i\} \subseteq V(G_1)$, and $(G_2, b_1, a_i, b_2, c_3, c_2, c_1)$ is planar.

(R5) There exists a separation (G_1, G_2) in G of order 4, satisfying one of the following properties:

(a) let W be a graph with $V(W) = \{w_0, w_1, w_2, w_3, w_4\}$, $E(W) = \{w_0 w_i; i = 1, 2, 3, 4\} \cup \{w_1 w_2, w_1 w_3\}$, then $a_0, a_1, a_2, b_1, b_2 \in V(G_1)$, $V(G_2 - G_1) \neq \emptyset$, and G_2 is not a subgraph of W; or

(b) $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}$, $a_0, a_1, a_2, b_1, b_2 \in V(G_1)$, $V(G_2 - G_1) = \{c\}$, G has edges from c to c_1, c_2, c_3, c_4, G has edges from c_1 to c_2, c_3, and for some $i \in \{0, 1, 2\}$ and some $j \in \{1, 2\}$, $a_i, b_j \in V(G_1 \cap G_2)$; or

(c) for some $i \in \{0, 1, 2\}$ and some $j \in \{1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, a_i, b_j\}$, $a_0, a_1, a_2, b_1, b_2 \in V(G_1)$, $V(G_2 - G_1) = \{c\}$, G has edges from c to c_1, c_2, a_i, b_j, and G has an edge from c_1 to c_2; or

(d) $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}$, $a_0, a_1, a_2, b_1, b_2 \in V(G_1)$, $V(G_2 - G_1) = \{c\}$, G has edges from c to c_1, c_2, c_3, c_4, G has an edge from c_1 to c_2, and for some permutation π of $\{0, 1, 2\}$, $\{a_{\pi(0)}, a_{\pi(1)}\} \subseteq V(G_1 \cap G_2)$ and $\{a_{\pi(0)}, a_{\pi(1)}\} \cap \{c_1, c_2\} \neq \emptyset$; or

(e) for some $i \in \{0, 1, 2\}$, $\{a_i\} = V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}$, $V(G_1 \cap G_2) = \{b_1, b_2, c_1, c_2\}$, $(G_2, a_i, b_1, c_1, c_2, b_2)$ is planar, and $|V(G_2 - G_1)| \geq 2$; or

(f) for some permutation π of $\{0, 1, 2\}$ and some $j \in \{1, 2\}$, $\{b_j\} = V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}$, $V(G_1 \cap G_2) = \{a_{\pi(1)}, a_{\pi(2)}, c_1, c_2\}$, $(G_2, b_j, a_{\pi(1)}, c_1, c_2, a_{\pi(2)})$ is planar, and $|V(G_2 - G_1)| \geq 2$; or

(g) for some permutation π of $\{0, 1, 2\}$ and some $j \in \{1, 2\}$, $\{a_{\pi(0)}\} = V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}$, $V(G_1 \cap G_2) = \{b_j, a_{\pi(1)}, c_1, c_2\}$, $(G_2, a_{\pi(0)}, b_j, c_1, a_{\pi(1)}, c_2)$ is planar, and $|V(G_2 - G_1)| \geq 2$; or

(h) for some permutation π of $\{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3, a_{\pi(0)}\}$, $a_{\pi(1)}, b_j \in V(G_2 - G_1)$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, $(G_2, c_1, c_2, a_{\pi(0)}, c_3, a_{\pi(1)}, b_j)$ is planar, and $|V(G_2 - G_1)| \geq 3$; or

(i) for some permutation π of $\{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3, a_{\pi(0)}\}$, $a_{\pi(1)}, b_j \in V(G_2 - G_1)$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, $(G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_3, c_2, c_1)$ is planar, and $|V(G_2 - G_1)| \geq 3$; or

(j) for some $i \in \{0, 1, 2\}$ and some $j \in \{1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3, b_j\}$, $a_i, b_{3-j} \in V(G_2 - G_1)$, $\{a_1, a_2, a_3\} - a_i \subseteq V(G_1)$, $(G_2, b_{3-j}, a_i, b_j, c_3, c_2, c_1)$ is planar, and $|V(G_2 - G_1)| \geq 3$; or

(k) for some permutation π of $\{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}$, $a_{\pi(0)}, a_{\pi(1)}, b_j \in V(G_2 - G_1)$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, $(G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_4, c_3, c_2, c_1)$ is planar, and $|V(G_2 - G_1)| \geq 4$; or

(l) $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}$, $a_i, b_1, b_2 \in V(G_2 - G_1)$, $\{a_1, a_2, a_3\} - a_i \subseteq V(G_1)$, $(G_2, b_1, a_i, b_2, c_4, c_3, c_2, c_1)$ is planar, and $|V(G_2 - G_1)| \geq 4$; or

(m) for some permutation π of $\{0,1,2\}$, $a_{\pi(0)}, a_{\pi(1)}, b_1, b_2 \in V(G_1)$, $\{a_{\pi(0)}, a_{\pi(1)}, b_1, b_2\} \cap V(G_2) \neq \emptyset$, $a_{\pi(2)} \in V(G_2) - V(G_1)$, and G_1 has a disk representation in which $a_{\pi(0)}, b_1, a_{\pi(1)}, b_2$ occur on the boundary of the disk in the order listed and the vertices in $V(G_1) \cap V(G_2)$ are incident with a common finite face.

(R6) There exists a separation (G_1, G_2) in G of order 5, satisfying one of the following properties:

(a) $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4, c_5\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $E(G[\{c_1, c_2, c_3, c_4, c_5\}]) \subseteq E(G_1)$, $(G_2, c_1, c_2, c_3, c_4, c_5)$ is planar, and $|V(G_2 - G_1)| \geq 2$; or

(b) $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4, c_5\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, and for some permutation π of $\{0,1,2\}$, G_1 has a disk representation with the vertices $a_{\pi(0)}, b_1, a_{\pi(1)}, b_2, a_{\pi(2)}, c_1, c_2, c_3, c_4, c_5$ drawn on the boundary of the disk in the order listed; or

(c) for some permutation π of $\{0,1,2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, b_1, b_2, a_{\pi(1)}\}$, $a_{\pi(2)} \in V(G_1 - G_2)$, $a_{\pi(0)} \in V(G_2 - G_1)$, $(G_2, b_1, c_1, a_{\pi(1)}, c_2, b_2, a_{\pi(0)})$ is planar, and $|V(G_2 - G_1)| \geq 4$; or

(d) for some $j \in \{1,2\}$ and some permutation π of $\{0,1,2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3, a_{\pi(1)}, b_j\}$, $a_{\pi(2)} \in V(G_1 - G_2)$, $a_{\pi(0)}, b_{3-j} \in V(G_2 - G_1)$, $(G_2, a_{\pi(1)}, c_1, c_2, c_3, b_j, a_{\pi(0)}, b_{3-j})$ is planar, and $|V(G_2 - G_1)| \geq 3$.

Actually, we can prove that if $(G, a_0, a_1, a_2, b_1, b_2)$ is reducible, then we could either easily determine whether or not $(G, a_0, a_1, a_2, b_1, b_2)$ is feasible, or reduce $(G, a_0, a_1, a_2, b_1, b_2)$ to $(G', a_0', a_1', a_2', b_1', b_2')$ with $(|V(G)|, |E(G)|) > (|V(G')|, |E(G')|)$ in lexicographic order, such that $(G, a_0, a_1, a_2, b_1, b_2)$ is feasible iff $(G', a_0', a_1', a_2', b_1', b_2')$ is feasible.

With all these, we can state our main result.

Theorem 7.0.1 *Let $(G, a_0, a_1, a_2, b_1, b_2)$ be a rooted graph. Then one of the following conclusions holds:*

(C1) *There exists a cluster* $\{X_1, X_2\}$ *in* G *such that* $\{a_0, a_1, a_2\} \subseteq X_1$ *and* $\{b_1, b_2\} \subseteq X_2$.

(C2) $(G, a_0, a_1, a_2, b_1, b_2)$ *is reducible.*

(C3) *For some* $i \in \{0, 1, 2\}$, $G - a_i$ *has no cluster* $\{X_1, X_2\}$ *such that* $\{a_0, a_1, a_2\} - \{a_i\} \subseteq X_1$ *and* $\{b_1, b_2\} \subseteq X_2$.

(C4) *There exist a permutation* π *of* $\{0, 1, 2\}$*, a graph* H *and vertices* $s, t, s', t' \in V(H)$ *such that* G *is obtained from* H *by identifying* s *with* s' *and* t *with* t'*, respectively, and* H *has a disk representation with the vertices* $a_{\pi(0)}, b_1, a_{\pi(1)}, b_2, a_{\pi(2)}, s, t, s', t'$ *drawn on the boundary of the disk in the order listed.*

(C5) G *has a separation* (G_1, G_2) *in* G *of order 4, such that* $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}$, $a_0, a_1, a_2, b_1, b_2 \in V(G_1)$*, and there exist a permutation* π *of* $\{0, 1, 2\}$*, a graph* H *and vertices* $c_2', c_2'' \in V(H)$*, where* G_1 *is obtained from* H *by identifying* c_2' *with* c_2'', $(H, a_{\pi(1)}, b_1, a_{\pi(0)}, b_2, a_{\pi(2)}, c_2'', c_4, c_3, c_2', c_1)$ *is planar, and* $c_2 \in V(G_1)$ *is the vertex obtained by identifying* c_2' *with* c_2''.

7.0.2 Clarifying (C3)

Note that if (C4) or (C5) holds, then (C1) will not hold. However, if (C3) holds, $(G, a_0, a_1, a_2, b_1, b_2)$ may be feasible or may be infeasible. Although by using 2-linkage algorithms, it is easy to judge whether $(G, a_0, a_1, a_2, b_1, b_2)$ admits (C3), we want to give a more precise characterization of feasible rooted graphs when (C3) holds.

We will still assume G is not reducible. So by applying Seymour's version of 2-linkage theorem in [37], when (C3) holds, there exists $i \in \{0, 1, 2\}$, such that $(G - a_i, a_{i+1}, b_1, a_{i-1}, b_2)$ is planar. So G actually is an apex graph.

7.0.3 A faster algorithm

Another possible future work is to develop a faster polynomial time algorithm for the Two-Three Linkage Problem.

Note that the existence of such an algorithm with polynomial running time is guaranteed by the work of Robertson and Seymour in [40]: Given a graph G and $k \geq 1$ pairs of vertices $\{s_i, t_i\}$, $i = 1, \cdots, k$ of G with k fixed, there exists a polynomial time algorithm for deciding if there are k mutually internally vertex-disjoint paths in G joining s_i and t_i, $i = 1, \cdots, k$. In fact, to resolve the Two-Three Linkage Problem, we just need to check:

(i) whether for some $i \in \{0, 1, 2\}$, G contains 3 mutually internally vertex-disjoint paths joining the pairs $\{b_1, b_2\}, \{a_{i-1}, a_i\}$ and $\{a_i, a_{i+1}\}$; or

(ii) whether for some vertex $v \in V(G) - \{a_0, a_1, a_2, b_1, b_2\}$, G contains 4 mutually vertex-disjoint paths to join the pairs $\{b_1, b_2\}, \{v, a_0\}, \{v, a_1\}$ and $\{v, a_2\}$.

Clearly, the answer is yes iff $(G, a_0, a_1, a_2, b_1, b_2)$ is feasible. The disjoint paths algorithm of Robertson and Seymour has running time $O(|V(G)|^3)$. So the above algorithm runs $O(|V(G)|^4)$ time.

However, the disjoint paths algorithm of Robertson and Seymour is not practical, since it involves an enormous constant. Hence, it is meaningful to come up with a faster algorithm for the two-three linkage problem. In fact, to the best of our knowledge, Tholey [41] found the $O(m + n\alpha(n, n))$-time algorithm, the currently best known nearly linear time bound, of 2-linkage problem, where α denotes the inverse of the Ackermann function. By repeatedly using 2-linkage algorithm, we expect to obtain a $O(|V(G)|^3)$-time two-three linkage algorithm.

7.0.4 Related conjecture

A graph G is apex if $G - v$ is planar for some vertex $v \in V(G)$. Jørgensen [34] conjectured that every 6-connected graph with no K_6-minor is apex.

In the two-three linkage problem, we only consider finding disjoint connected subgraphs G_1, G_2 such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$ and $\{b_1, b_2\} \subseteq V(G_2)$. However, it is also natural to ask whether we can find such disjoint connected subgraphs G_1, G_2 satisfying additional properties. For example, we have

Conjecture 7.0.2 *Any 6-connected non-apex graph G with distinct vertices* $a_0, a_1, a_2, b_1, b_2 \in V(G)$ *contains disjoint connected subgraphs* G_1, G_2 *such that* $\{a_0, a_1, a_2\} \subseteq V(G_1)$*,* $\{b_1, b_2\} \subseteq V(G_2)$*, and the following properties hold:*

(P1) there exists a vertex $v \in G_1 - \{a_0, a_1, a_2\}$ *such that* G_1 *has three disjoint paths from* v *to* a_0, a_1, a_2*, respectively;*

(P2) for each vertex $v \in G_1$*, the vertices* a_0, a_1, a_2 *are contained in one component of* $G_1 - v$*.*

One observation is that if there exists $v \in V(G)$ such that $(G-v, a_1, b_1, a_2, b_2)$ is planar, then there do not exist disjoint connected subgraphs G_1, G_2 in G such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$, $\{b_1, b_2\} \subseteq V(G_2)$, and G_1 satisfies (P1) and (P2). Note that such G is apex, and G can be 6-connected.

If Conjecture 7.0.2 is true, we may prove that given a 6-connected graph G and triangles $a_i b_1 b_2 a_i$ for $i = 0, 1, 2$, $G - b_1 b_2 - \{a_i b_j : i = 0, 1, 2 \text{ and } j = 1, 2\}$ contains disjoint connected subgraphs G_1, G_2 such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$, $\{b_1, b_2\} \subseteq V(G_2)$, and G_1 satisfies (P1) and (P2). Such properties could be useful in resolving Jørgensen's conjecture for 6-connected graph in which some edge is contained in three triangles.

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